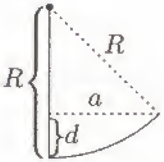
Chapter 5

Magnetostatics

Problem 5.1

Since $\mathbf{v} \times \mathbf{B}$ points upward, and that is also the direction of the force, q must be $\boxed{\text{positive.}}$ To find R, in terms of a and d, use the pythagorean theorem:

$$(R-d)^2 + a^2 = R^2 \;\Rightarrow\; R^2 - 2Rd + d^2 + a^2 = R^2 \;\Rightarrow R = \frac{a^2+d^2}{2d}.$$



The cyclotron formula then gives

$$p = qBR = \boxed{qB\frac{(a^2+d^2)}{2d}.}$$

Problem 5.2

The general solution is (Eq. 5.6):

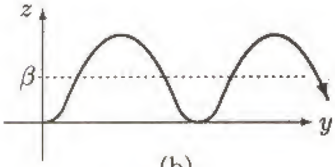
$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{E}{B}t + C_3; \quad z(t) = C_2\cos(\omega t) - C_1\sin(\omega t) + C_4.$$

(a) $y(0) = z(0) = 0$; $\dot{y}(0) = E/B$; $\dot{z}(0) = 0$. Use these to determine C_1, C_2, C_3, and C_4.
$y(0) = 0 \Rightarrow C_1 + C_3 = 0$; $\dot{y}(0) = \omega C_2 + E/B = E/B \Rightarrow C_2 = 0$; $z(0) = 0 \Rightarrow C_2 + C_4 = 0 \Rightarrow C_4 = 0$; $\dot{z}(0) = 0 \Rightarrow C_1 = 0$, and hence also $C_3 = 0$. So $\boxed{y(t) = Et/B;\; z(t) = 0.}$ Does this make sense? The magnetic force is $q(\mathbf{v} \times \mathbf{B}) = -q(E/B)B\,\hat{\mathbf{z}} = -q\mathbf{E}$, which exactly cancels the electric force; since there is no net force, the particle moves in a straight line at constant speed. ✓

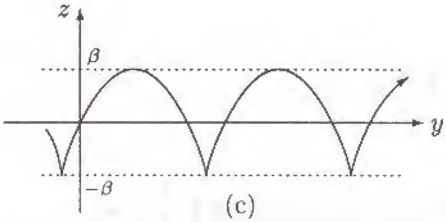
(b) Assuming it starts from the origin, so $C_3 = -C_1$, $C_4 = -C_2$, we have $\dot{z}(0) = 0 \Rightarrow C_1 = 0 \Rightarrow C_3 = 0$; $\dot{y}(0) = \frac{E}{2B} \Rightarrow C_2\omega + \frac{E}{B} = \frac{E}{2B} \Rightarrow C_2 = -\frac{E}{2\omega B} = -C_4$; $y(t) = -\frac{E}{2\omega B}\sin(\omega t) + \frac{E}{B}t$; $z(t) = -\frac{E}{2\omega B}\cos(\omega t) + \frac{E}{2\omega B}$, or $\boxed{y(t) = \frac{E}{2\omega B}\left[2\omega t - \sin(\omega t)\right];\; z(t) = \frac{E}{2\omega B}\left[1 - \cos(\omega t)\right].}$ Let $\beta \equiv E/2\omega B$. Then $y(t) = \beta\left[2\omega t - \sin(\omega t)\right]$; $z(t) = \beta\left[1 - \cos(\omega t)\right]$; $(y - 2\beta\omega t) = -\beta\sin(\omega t)$, $(z - \beta) = -\beta\cos(\omega t) \Rightarrow$ $(y - 2\beta\omega t)^2 + (z-\beta)^2 = \beta^2$. This is a circle of radius β whose center moves to the right at constant speed: $y_0 = 2\beta\omega t$; $z_0 = \beta$.

(c) $\dot{z}(0) = \dot{y}(0) = \frac{E}{B} \Rightarrow -C_1\omega = \frac{E}{B} \Rightarrow C_1 = -C_3 = -\frac{E}{\omega B}$; $C_2\omega + \frac{E}{B} = \frac{E}{B} \Rightarrow C_2 = C_4 = 0$.

$y(t) = -\frac{E}{\omega B}\cos(\omega t) + \frac{E}{B}t + \frac{E}{\omega B}$; $z(t) = \frac{E}{\omega B}\sin(\omega t)$. $\boxed{y(t) = \frac{E}{\omega B}\left[1 + \omega t - \cos(\omega t)\right];\ z(t) = \frac{E}{\omega B}\sin(\omega t).}$
Let $\beta \equiv E/\omega B$; then $[y - \beta(1+\omega t)] = -\beta\cos(\omega t)$, $z = \beta\sin(\omega t)$; $[y - \beta(1+\omega t)]^2 + z^2 = \beta^2$. This is a circle of radius β whose center is at $y_0 = \beta(1+\omega t)$, $z_0 = 0$.



(b)



(c)

Problem 5.3

(a) From Eq. 5.2, $\mathbf{F} = q[\mathbf{E} + (\mathbf{v}\times\mathbf{B})] = 0 \Rightarrow E = vB \Rightarrow \boxed{v = \frac{E}{B}.}$

(b) From Eq. 5.3, $mv = qBR \Rightarrow \frac{q}{m} = \frac{v}{BR} = \boxed{\frac{E}{B^2R}.}$

Problem 5.4

Suppose I flows counterclockwise (if not, change the sign of the answer). The force on the left side (toward the left) cancels the force on the right side (toward the right); the force on the top is $IaB = Iak(a/2) = Ika^2/2$, (pointing upward), and the force on the bottom is $IaB = -Ika^2/2$ (also upward). So the net force is $\mathbf{F} = \boxed{Ika^2\,\hat{\mathbf{z}}.}$

Problem 5.5

(a) $\boxed{K = \frac{I}{2\pi a},}$ because the length-perpendicular-to-flow is the circumference.

(b) $J = \frac{\alpha}{s} \Rightarrow I = \int J\,da = \alpha\int\frac{1}{s}s\,ds\,d\phi = 2\pi\alpha\int ds = 2\pi\alpha a \Rightarrow \alpha = \frac{I}{2\pi a}$; $J = \boxed{\frac{I}{2\pi as}.}$

Problem 5.6

(a) $v = \omega r$, so $\boxed{K = \sigma\omega r.}$ (b) $\mathbf{v} = \omega r\sin\theta\,\hat{\boldsymbol{\phi}} \Rightarrow \boxed{\mathbf{J} = \rho\omega r\sin\theta\,\hat{\boldsymbol{\phi}},}$ where $\rho \equiv Q/(4/3)\pi R^3$.

Problem 5.7

$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\int_{\mathcal{V}}\rho\mathbf{r}\,d\tau = \int\left(\frac{\partial\rho}{\partial t}\right)\mathbf{r}\,d\tau = -\int(\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{r}\,d\tau$ (by the continuity equation). Now product rule #5 says $\boldsymbol{\nabla}\cdot(x\mathbf{J}) = x(\boldsymbol{\nabla}\cdot\mathbf{J}) + \mathbf{J}\cdot(\boldsymbol{\nabla}x)$. But $\boldsymbol{\nabla}x = \hat{\mathbf{x}}$, so $\boldsymbol{\nabla}\cdot(x\mathbf{J}) = x(\boldsymbol{\nabla}\cdot\mathbf{J}) + J_x$. Thus $\int_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\mathbf{J})x\,d\tau = \int_{\mathcal{V}}\boldsymbol{\nabla}\cdot(x\mathbf{J})\,d\tau - \int_{\mathcal{V}}J_x\,d\tau$. The first term is $\int_{\mathcal{S}}x\mathbf{J}\cdot d\mathbf{a}$ (by the divergence theorem), and since $\mathbf{J}$ is entirely *inside* $\mathcal{V}$, it is zero on the surface $\mathcal{S}$. Therefore $\int_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\mathbf{J})x\,d\tau = -\int_{\mathcal{V}}J_x\,d\tau$, or, combining this with the y and z components, $\int_{\mathcal{V}}(\boldsymbol{\nabla}\cdot\mathbf{J})\mathbf{r}\,d\tau = -\int_{\mathcal{V}}\mathbf{J}\,d\tau$. Or, referring back to the first line, $\frac{d\mathbf{p}}{dt} = \int\mathbf{J}\,d\tau$. qed

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Problem 5.8

(a) Use Eq. 5.35, with $z = R, \theta_2 = -\theta_1 = 45°$, and four sides: $B = \boxed{\frac{\sqrt{2}\mu_0 I}{\pi R}.}$

(b) $z = R$, $\theta_2 = -\theta_1 = \frac{\pi}{n}$, and n sides: $B = \boxed{\frac{n\mu_0 I}{2\pi R}\sin(\pi/n).}$

(c) For small θ, $\sin\theta \approx \theta$. So as $n \to \infty, B \to \frac{n\mu_0 I}{2\pi R}\left(\frac{\pi}{n}\right) = \boxed{\frac{\mu_0 I}{2R}}$ (same as Eq. 5.38, with $z = 0$).

Problem 5.9

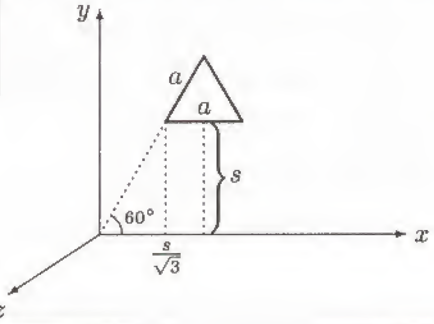
(a) The straight segments produce no field at P. The two quarter-circles give $B = \boxed{\frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)}$ (out).

(b) The two half-lines are the same as one infinite line: $\frac{\mu_0 I}{2\pi R}$; the half-circle contributes $\frac{\mu_0 I}{4R}$.
So $B = \boxed{\frac{\mu_0 I}{4R}\left(1 + \frac{2}{\pi}\right)}$ (into the page).

Problem 5.10

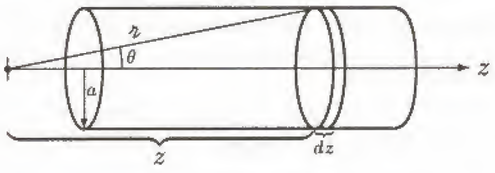
(a) The forces on the two sides cancel. At the bottom, $B = \frac{\mu_0 I}{2\pi s} \Rightarrow F = \left(\frac{\mu_0 I}{2\pi s}\right) Ia = \frac{\mu_0 I^2 a}{2\pi s}$ (up). At the top, $B = \frac{\mu_0 I}{2\pi(s+a)} \Rightarrow F = \frac{\mu_0 I^2 a}{2\pi(s+a)}$ (down). The net force is $\boxed{\frac{\mu_0 I^2 a^2}{2\pi s(s+a)}}$ (up).

(b) The force on the bottom is the same as before, $\mu_0 I^2/2\pi$ (up). On the left side, $\mathbf{B} = \frac{\mu_0 I}{2\pi y}\,\hat{\mathbf{z}}$; $d\mathbf{F} = I(d\mathbf{l} \times \mathbf{B}) = I(dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}) \times \left(\frac{\mu_0 I}{2\pi y}\,\hat{\mathbf{z}}\right) = \frac{\mu_0 I^2}{2\pi y}(-dx\,\hat{\mathbf{y}} + dy\,\hat{\mathbf{x}})$. But the x component cancels the corresponding term from the right side, and $F_y = -\frac{\mu_0 I^2}{2\pi}\int_{s/\sqrt{3}}^{(s/\sqrt{3}+a/2)} \frac{1}{y}\,dx$. Here $y = \sqrt{3}x$, so $F_y = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(\frac{s/\sqrt{3}+a/2}{s/\sqrt{3}}\right) = -\frac{\mu_0 I^2}{2\sqrt{3}\pi}\ln\left(1 + \frac{\sqrt{3}a}{2s}\right)$. The force on the right side is the same, so the net force on the triangle is $\boxed{\frac{\mu_0 I^2}{2\pi}\left[1 - \frac{2}{\sqrt{3}}\ln\left(1 + \frac{\sqrt{3}a}{2s}\right)\right]}$.



Problem 5.11

Use Eq. 5.38 for a ring of width dz, with $I \to nI\,dz$:
$B = \frac{\mu_0 nI}{2}\int \frac{a^2}{(a^2+z^2)^{3/2}}\,dz$. But $z = a\cot\theta$,
so $dz = -\frac{a}{\sin^2\theta}\,d\theta$, and $\frac{1}{(a^2+z^2)^{3/2}} = \frac{\sin^3\theta}{a^3}$.
So



$$B = \frac{\mu_0 nI}{2}\int \frac{a^2\sin^3\theta}{a^3\sin^2\theta}(-a\,d\theta) = -\frac{\mu_0 nI}{2}\int \sin\theta\,d\theta = \frac{\mu_0 nI}{2}\cos\theta\Big|_{\theta_1}^{\theta_2} = \boxed{\frac{\mu_0 nI}{2}(\cos\theta_2 - \cos\theta_1).}$$

For an infinite solenoid, $\theta_2 = 0$, $\theta_1 = \pi$, so $(\cos\theta_2 - \cos\theta_1) = 1 - (-1) = 2$, and $B = \boxed{\mu_0 nI.}$ ✓

Problem 5.12

Magnetic attraction per unit length (Eqs. 5.37 and 5.13): $f_m = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$.

Electric field of one wire (Eq. 2.9): $E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}$. Electric repulsion per unit length on the other wire: $f_e = \frac{1}{2\pi\epsilon_0}\frac{\lambda^2}{d}$. They balance when $\mu_0 v^2 = \frac{1}{\epsilon_0}$, or $\boxed{v = \frac{1}{\sqrt{\epsilon_0\mu_0}}.}$ Putting in the numbers,

$v = \frac{1}{\sqrt{(8.85\times 10^{-12})(4\pi\times 10^{-7})}} = \boxed{3.00\times 10^8\,\mathrm{m/s}.}$ This is precisely the *speed of light(!)*, so in fact you could *never* get the wires going fast enough; the electric force always dominates.

Problem 5.13

(a) $\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 I_{\text{enc}} \Rightarrow \boxed{\mathbf{B} = \left\{\begin{array}{ll} 0, & \text{for } s < a; \\ \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

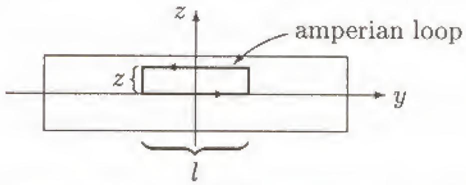
(b) $J = ks$; $I = \int_0^a J\,da = \int_0^a ks(2\pi s)\,ds = \frac{2\pi k a^3}{3} \Rightarrow k = \frac{3I}{2\pi a^3}$. $I_{\text{enc}} = \int_0^s J\,da = \int_0^s k\bar{s}(2\pi\bar{s})\,d\bar{s} = \frac{2\pi k s^3}{3} = I\frac{s^3}{a^3}$, for $s < a$; $I_{\text{enc}} = I$, for $s > a$. So $\boxed{\mathbf{B} = \left\{\begin{array}{ll} \frac{\mu_0 I s^2}{2\pi a^3}\,\hat{\boldsymbol{\phi}}, & \text{for } s < a; \\ \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}, & \text{for } s > a. \end{array}\right\}}$

Problem 5.14

By the right-hand-rule, the field points in the $-\hat{\mathbf{y}}$ direction for $z > 0$, and in the $+\hat{\mathbf{y}}$ direction for $z < 0$. At $z = 0, B = 0$. Use the amperian loop shown:

$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{\text{enc}} = \mu_0 l z J \Rightarrow \boxed{\mathbf{B} = -\mu_0 J z\,\hat{\mathbf{y}}}$ $(-a < z < a)$. If $z > a, I_{\text{enc}} = \mu_0 l a J$,

so $\boxed{\mathbf{B} = \left\{\begin{array}{ll} -\mu_0 J a\,\hat{\mathbf{y}}, & \text{for } z > +a; \\ +\mu_0 J a\,\hat{\mathbf{y}}, & \text{for } z > -a. \end{array}\right\}}$



Problem 5.15

The field inside a solenoid is $\mu_0 n I$, and outside it is zero. The outer solenoid's field points to the left $(-\hat{\mathbf{z}})$, whereas the inner one points to the right $(+\hat{\mathbf{z}})$. So: (i) $\boxed{\mathbf{B} = \mu_0 I(n_1 - n_2)\,\hat{\mathbf{z}},}$ (ii) $\boxed{\mathbf{B} = -\mu_0 I n_2\,\hat{\mathbf{z}},}$ (iii) $\boxed{\mathbf{B} = 0.}$

Problem 5.16

From Ex. 5.8, the top plate produces a field $\mu_0 K/2$ (aiming *out of the page*, for points above it, and *into the page*, for points below). The bottom plate produces a field $\mu_0 K/2$ (aiming *into the page*, for points above it, and *out of the page*, for points below). Above and below *both* plates the two fields cancel; *between* the plates they add up to $\mu_0 K$, pointing *in*.

(a) $\boxed{B = \mu_0\sigma v \text{ (in)}}$ betweem the plates, $\boxed{B = 0}$ elsewhere.

(b) The Lorentz force law says $\mathbf{F} = \int(\mathbf{K}\times\mathbf{B})\,da$, so the force *per unit area* is $\mathbf{f} = \mathbf{K}\times\mathbf{B}$. Here $K = \sigma v$, to the right, and $\mathbf{B}$ (the field of the lower plate) is $\mu_0\sigma v/2$, into the page. So $\boxed{f_m = \mu_0\sigma^2 v^2/2 \text{ (up)}.}$

(c) The electric field of the lower plate is $\sigma/2\epsilon_0$; the electric force per unit area on the upper plate is $\boxed{f_e = \sigma^2/2\epsilon_0 \text{ (down)}.}$ They balance if $\mu_0 v^2 = 1/\epsilon_0$, or $\boxed{v = 1/\sqrt{\epsilon_0\mu_0} = c}$ (the speed of light), as in Prob. 5.12.

Problem 5.17

We might as well orient the axes so the field point $\mathbf{r}$ lies on the y axis: $\mathbf{r} = (0, y, 0)$. Consider a source point at (x', y', z') on loop #1:

$$\boldsymbol{\mathscr{r}} = -x'\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} - z'\,\hat{\mathbf{z}};\;\; d\mathbf{l}' = dx'\,\hat{\mathbf{x}} + dy'\,\hat{\mathbf{y}};$$

$$d\mathbf{l}' \times \boldsymbol{\mathscr{r}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ dx' & dy' & 0 \\ -x' & (y-y') & -z' \end{vmatrix} = (-z'\,dy')\,\hat{\mathbf{x}} + (z'\,dx')\,\hat{\mathbf{y}} + [(y-y')\,dx' + x'\,dy']\,\hat{\mathbf{z}}.$$

$$d\mathbf{B}_1 = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l}' \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi}\frac{(-z'\,dy')\,\hat{\mathbf{x}} + (z'\,dx')\,\hat{\mathbf{y}} + [(y-y')\,dx' + x'\,dy']\,\hat{\mathbf{z}}}{[(x')^2 + (y-y')^2 + (z')^2]^{3/2}}.$$

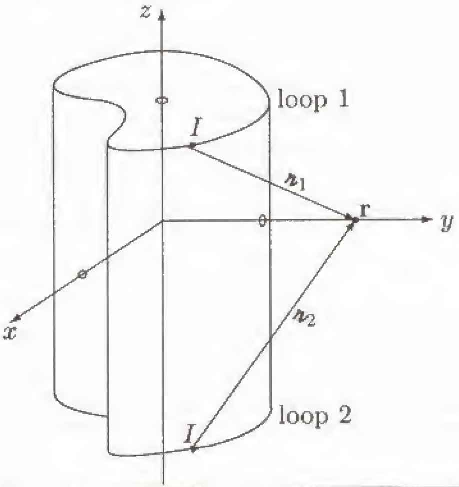
Now consider the symmetrically placed source element on loop #2, at $(x', y', -z')$. Since z' changes sign, while everything else is the same, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components from $d\mathbf{B}_1$ and $d\mathbf{B}_2$ cancel, leaving only a $\hat{\mathbf{z}}$ component. qed

With this, Ampére's law yields immediately:

$$\boxed{\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & \text{inside the solenoid;} \\ 0, & \text{outside} \end{cases}}$$

(the same as for a circular solenoid—Ex. 5.9).

For the toroid, $N/2\pi s = n$ (the number of turns per unit length), so Eq. 5.58 yields $B = \mu_0 n I$ inside, and zero outside, consistent with the solenoid. [*Note:* $N/2\pi s = n$ applies only if the toroid is large in circumference, so that s is essentially constant over the cross-section.]



Problem 5.18

$\boxed{\text{It doesn't matter.}}$ According to Theorem 2, in Sect. 1.6.2, $\int \mathbf{J}\cdot d\mathbf{a}$ is independent of surface, for any given boundary line, provided that $\mathbf{J}$ is divergenceless, which it *is*, for steady currents (Eq. 5.31).

Problem 5.19

(a) $\rho = \dfrac{\text{charge}}{\text{volume}} = \dfrac{\text{charge}}{\text{atom}}\cdot\dfrac{\text{atoms}}{\text{mole}}\cdot\dfrac{\text{moles}}{\text{gram}}\cdot\dfrac{\text{grams}}{\text{volume}} = (e)(N)\left(\dfrac{1}{M}\right)(d)$, where

$$\begin{aligned} e &= \text{charge of electron} &&= 1.6\times10^{-19}\,\text{C}, \\ N &= \text{Avogadro's number} &&= 6.0\times10^{23}\,\text{mole}, \\ M &= \text{atomic mass of copper} &&= 64\,\text{gm/mole}, \\ d &= \text{density of copper} &&= 9.0\,\text{gm/cm}^3. \end{aligned}$$

$$\rho = (1.6\times10^{-19})(6.0\times10^{23})\left(\frac{9.0}{64}\right) = \boxed{1.4\times10^4\,\text{C/cm}^3.}$$

(b) $J = \dfrac{I}{\pi s^2} = \rho v \Rightarrow v = \dfrac{I}{\pi s^2 \rho} = \dfrac{1}{\pi(2.5\times10^{-3})(1.4\times10^4)} = \boxed{9.1\times10^{-3}\,\text{cm/s},}$ or about 33 cm/hr. This is astonishingly small—literally slower than a snail's pace.

(c) From Eq. 5.37, $f_m = \dfrac{\mu_0}{2\pi}\left(\dfrac{I_1 I_2}{d}\right) = \dfrac{(4\pi\times10^{-7})}{2\pi} = \boxed{2\times10^{-7}\,\text{N/cm}.}$

(d) $E = \dfrac{1}{2\pi\epsilon_0}\dfrac{\lambda}{d}$; $\quad f_e = \dfrac{1}{2\pi\epsilon_0}\left(\dfrac{\lambda_1\lambda_2}{d}\right) = \dfrac{1}{v^2}\dfrac{1}{2\pi\epsilon_0}\left(\dfrac{I_1 I_2}{d}\right) = \left(\dfrac{c^2}{v^2}\right)\dfrac{\mu_0}{2\pi}\left(\dfrac{I_1 I_2}{d}\right) = \dfrac{c^2}{v^2} f_m$, where $c \equiv 1/\sqrt{\epsilon_0\mu_0} = 3.00 \times 10^8$ m/s. Here $\dfrac{f_e}{f_m} = \dfrac{c^2}{v^2} = \left(\dfrac{3.0 \times 10^{10}}{9.1 \times 10^{-3}}\right)^2 = \boxed{1.1 \times 10^{25}.}$

$f_e = (1.1 \times 10^{25})(2 \times 10^{-7}) = \boxed{2 \times 10^{18}\ \text{N/cm.}}$

Problem 5.20

Ampére's law says $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. Together with the continuity equation (5.29) this gives $\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J} = -\mu_0 \partial\rho/\partial t$, which is *inconsistent* with div(curl)=0 unless ρ is constant (magnetostatics). The other Maxwell equations are OK: $\nabla \times \mathbf{E} = 0 \Rightarrow \nabla \cdot (\nabla \times \mathbf{E}) = 0$ (✓), and as for the two divergence equations, there is no relevant vanishing second derivative (the other one is curl(grad), which doesn't involve the divergence).

Problem 5.21

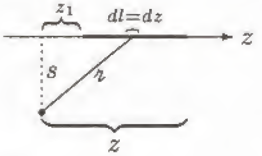
At this stage I'd expect no changes in Gauss's law or Ampére's law. The divergence of $\mathbf{B}$ would take the form $\boxed{\nabla \cdot \mathbf{B} = \alpha_0 \rho_m,}$ where ρ_m is the density of magnetic charge, and α_0 is some constant (analogous to ϵ_0 and μ_0). The curl of $\mathbf{E}$ becomes $\boxed{\nabla \times \mathbf{E} = \beta_0 \mathbf{J}_m,}$ where $\mathbf{J}_m$ is the magnetic current density (representing the flow of magnetic charge), and β_0 is another constant. Presumably magnetic charge is conserved, so ρ_m and $\mathbf{J}_m$ satisfy a continuity equation: $\nabla \cdot \mathbf{J}_m = -\partial\rho_m/\partial t$.

As for the Lorentz force law, one might guess something of the form $q_m[\mathbf{B} + (\mathbf{v} \times \mathbf{E})]$ (where q_m is the magnetic charge). But this is dimensionally impossible, since E has the same units as vB. Evidently we need to divide $(\mathbf{v} \times \mathbf{E})$ by something with the dimensions of velocity-squared. The natural candidate is $c^2 = 1/\epsilon_0\mu_0$: $\boxed{\mathbf{F} = q_e\left[\mathbf{E} + (\mathbf{v} \times \mathbf{B})\right] + q_m\left[\mathbf{B} - \dfrac{1}{c^2}(\mathbf{v} \times \mathbf{E})\right].}$ In this form the magnetic analog to Coulomb's law reads $\mathbf{F} = \dfrac{\alpha_0}{4\pi}\dfrac{q_{m_1} q_{m_2}}{r^2}\,\hat{\mathbf{r}}$, so to determine α_0 we would first introduce (arbitrarily) a unit of magnetic charge, then measure the force between unit charges at a given separation. [For further details, and an explanation of the minus sign in the force law, see Prob. 7.35.]

Problem 5.22

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{I\,\hat{\mathbf{z}}}{\mathscr{r}}\,dz = \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2 + s^2}}$$

$$= \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\left[\ln\left(z + \sqrt{z^2 + s^2}\right)\right]\Bigg|_{z_1}^{z_2} = \boxed{\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2 + \sqrt{(z_2)^2 + s^2}}{z_1 + \sqrt{(z_1)^2 + s^2}}\right]\hat{\mathbf{z}}}$$



$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A}{\partial s}\,\hat{\boldsymbol{\phi}} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2 + \sqrt{(z_2)^2 + s^2}}\frac{s}{\sqrt{(z_2)^2 + s^2}} - \frac{1}{z_1 + \sqrt{(z_1)^2 + s^2}}\frac{s}{\sqrt{(z_1)^2 + s^2}}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2 - \sqrt{(z_2)^2 + s^2}}{(z_2)^2 - [(z_2)^2 + s^2]}\frac{1}{\sqrt{(z_2)^2 + s^2}} - \frac{z_1 - \sqrt{(z_1)^2 + s^2}}{z_1^2 - [(z_1)^2 + s^2]}\frac{1}{\sqrt{(z_1)^2 + s^2}}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2 + s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2 + s^2}} + 1\right]\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2 + s^2}} - \frac{z_1}{\sqrt{(z_1)^2 + s^2}}\right]\hat{\boldsymbol{\phi}},$$

or, since $\sin\theta_1 = \dfrac{z_1}{\sqrt{(z_1)^2 + s^2}}$ and $\sin\theta_2 = \dfrac{z_2}{\sqrt{(z_2)^2 + s^2}}$,

$$= \boxed{\frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\,\hat{\boldsymbol{\phi}}} \text{ (as in Eq. 5.35).}$$

Problem 5.23

$A_\phi = k \Rightarrow \mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sk)\,\hat{\mathbf{z}} = \frac{k}{s}\,\hat{\mathbf{z}};\ \mathbf{J} = \frac{1}{\mu_0}(\nabla\times\mathbf{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\boldsymbol{\phi}} = \boxed{\frac{k}{\mu_0 s^2}\,\hat{\boldsymbol{\phi}}.}$

Problem 5.24

$\nabla\cdot\mathbf{A} = -\frac{1}{2}\nabla\cdot(\mathbf{r}\times\mathbf{B}) = -\frac{1}{2}[\mathbf{B}\cdot(\nabla\times\mathbf{r}) - \mathbf{r}\cdot(\nabla\times\mathbf{B})] = 0$, since $\nabla\times\mathbf{B} = 0$ ($\mathbf{B}$ is uniform) and $\nabla\times\mathbf{r} = 0$ (Prob. 1.62). $\nabla\times\mathbf{A} = -\frac{1}{2}\nabla\times(\mathbf{r}\times\mathbf{B}) = -\frac{1}{2}[(\mathbf{B}\cdot\nabla)\mathbf{r} - (\mathbf{r}\cdot\nabla)\mathbf{B} + \mathbf{r}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{r})]$. But $(\mathbf{r}\cdot\nabla)\mathbf{B} = 0$ and $\nabla\cdot\mathbf{B} = 0$ (since $\mathbf{B}$ is uniform), and $\nabla\cdot\mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$. Finally, $(\mathbf{B}\cdot\nabla)\mathbf{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) = B_x\,\hat{\mathbf{x}} + B_y\,\hat{\mathbf{y}} + B_z\,\hat{\mathbf{z}} = \mathbf{B}$. So $\nabla\times\mathbf{A} = -\frac{1}{2}(\mathbf{B} - 3\mathbf{B}) = \mathbf{B}$. qed

Problem 5.25

(a) $\mathbf{A}$ points in the same direction as $\mathbf{I}$, and is a function only of s (the distance from the wire). In cylindrical coordinates, then, $\mathbf{A} = A(s)\,\hat{\mathbf{z}}$, so $\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A}{\partial s}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$ (the field of an infinite wire). Therefore $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi s}$, and $\boxed{\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 I}{2\pi}\ln(s/a)\,\hat{\mathbf{z}}}$ (the constant a is arbitrary; you could use 1, but then the units look fishy). $\nabla\cdot\mathbf{A} = \frac{\partial A_z}{\partial z} = 0$. ✓ $\nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial s}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}} = \mathbf{B}$. ✓

(b) Here Ampére's law gives $\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0 J\,\pi s^2 = \mu_0\frac{I}{\pi R^2}\pi s^2 = \frac{\mu_0 I s^2}{R^2}$.

$\mathbf{B} = \frac{\mu_0}{2\pi}\frac{Is}{R^2}\,\hat{\boldsymbol{\phi}}$. $\frac{\partial A}{\partial s} = -\frac{\mu_0 I}{2\pi}\frac{s}{R^2} \Rightarrow \mathbf{A} = -\frac{\mu_0 I}{4\pi R^2}(s^2 - b^2)\,\hat{\mathbf{z}}$. Here b is again arbitrary, except that since $\mathbf{A}$ must be continuous at R, $-\frac{\mu_0 I}{2\pi}\ln(R/a) = -\frac{\mu_0 I}{4\pi R^2}(R^2 - b^2)$, which means that we must pick a and b such that $2\ln(R/b) = 1 - (b/R)^2$. I'll use $a = b = R$. Then

$$\boxed{\mathbf{A} = \left\{\begin{array}{ll} -\frac{\mu_0 I}{4\pi R^2}(s^2 - R^2)\,\hat{\mathbf{z}}, & \text{for } s \le R; \\ -\frac{\mu_0 I}{2\pi}\ln(s/R)\,\hat{\mathbf{z}}, & \text{for } s \ge R. \end{array}\right\}}$$

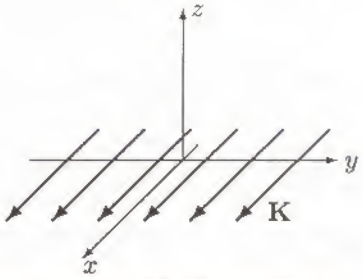
Problem 5.26

$\mathbf{K} = K\,\hat{\mathbf{x}} \Rightarrow \mathbf{B} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}$ (plus for $z < 0$, minus for $z > 0$).

$\mathbf{A}$ is parallel to $\mathbf{K}$, and depends only on z, so $\mathbf{A} = A(z)\,\hat{\mathbf{x}}$.

$$\mathbf{B} = \nabla\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A(z) & 0 & 0 \end{vmatrix} = \frac{\partial A}{\partial z}\,\hat{\mathbf{y}} = \pm\frac{\mu_0 K}{2}\,\hat{\mathbf{y}}.$$

$\boxed{\mathbf{A} = -\frac{\mu_0 K}{2}|z|\,\hat{\mathbf{x}}}$ will do the job—or this plus any constant.



Problem 5.27

(a) $\nabla\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau'$. $\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) + \mathbf{J}\cdot\nabla\left(\frac{1}{\mathscr{r}}\right)$. But the first term is zero, because $\mathbf{J}(\mathbf{r}')$ is a function of the *source* coordinates, not the *field* coordinates. And since $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$, $\nabla\left(\frac{1}{\mathscr{r}}\right) = -\nabla'\left(\frac{1}{\mathscr{r}}\right)$. So

$\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\mathbf{J}\cdot\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$. But $\boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\boldsymbol{\nabla}'\cdot\mathbf{J})+\mathbf{J}\cdot\boldsymbol{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$, and $\boldsymbol{\nabla}'\cdot\mathbf{J} = 0$ in magnetostatics (Eq. 5.31). So $\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)$, and hence, by the divergence theorem, $\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{\mu_0}{4\pi}\int \boldsymbol{\nabla}'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau' = -\frac{\mu_0}{4\pi}\oint \frac{\mathbf{J}}{\mathscr{r}}\cdot d\mathbf{a}'$, where the integral is now over the *surface* surrounding all the currents. But $\mathbf{J} = 0$ on this surface, so $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$. ✓

(b) $\boldsymbol{\nabla}\times\mathbf{A} = \frac{\mu_0}{4\pi}\int \boldsymbol{\nabla}\times\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau' = \frac{\mu_0}{4\pi}\int\left[\frac{1}{\mathscr{r}}(\boldsymbol{\nabla}\times\mathbf{J}) - \mathbf{J}\times\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right)\right]d\tau'$. But $\boldsymbol{\nabla}\times\mathbf{J} = 0$ (since $\mathbf{J}$ is not a function of $\mathbf{r}$), and $\boldsymbol{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$ (Eq. 1.101), so $\boldsymbol{\nabla}\times\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}d\tau' = \mathbf{B}$. ✓

(c) $\nabla^2\mathbf{A} = \frac{\mu_0}{4\pi}\int\nabla^2\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau'$. But $\nabla^2\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \mathbf{J}\nabla^2\left(\frac{1}{\mathscr{r}}\right)$ (once again, $\mathbf{J}$ is a *constant*, as far as differentiation with respect to $\mathbf{r}$ is concerned), and $\nabla^2\left(\frac{1}{\mathscr{r}}\right) = -4\pi\delta^3(\boldsymbol{\mathscr{r}})$ (Eq. 1.102).

So $\nabla^2\mathbf{A} = \frac{\mu_0}{4\pi}\int\mathbf{J}(\mathbf{r}')\left[-4\pi\delta^3(\boldsymbol{\mathscr{r}})\right]d\tau' = -\mu_0\mathbf{J}(\mathbf{r})$. ✓

Problem 5.28

$\mu_0 I = \oint\mathbf{B}\cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}}\boldsymbol{\nabla}U\cdot d\mathbf{l} = -[U(\mathbf{b}) - U(\mathbf{a})]$ (by the gradient theorem), so $U(\mathbf{b}) \neq U(\mathbf{a})$. qed

For an infinite straight wire, $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$. $\boxed{U = -\frac{\mu_0 I\phi}{2\pi}}$ would do the job, in the sense that $-\boldsymbol{\nabla}U = \frac{\mu_0 I}{2\pi}\boldsymbol{\nabla}(\phi) = \frac{\mu_0 I}{2\pi}\frac{1}{s}\frac{\partial\phi}{\partial\phi}\hat{\boldsymbol{\phi}} = \mathbf{B}$. But when ϕ advances by 2π, this function does *not* return to its initial value; it works (say) for $0 \leq \phi < 2\pi$, but at 2π it "jumps" back to zero.

Problem 5.29

Use Eq. 5.67, with $R \to \bar{r}$ and $\sigma \to \rho\, d\bar{r}$:

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0\omega\rho}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\phi}}\int_0^r \bar{r}^4\,d\bar{r} + \frac{\mu_0\omega\rho}{3}r\sin\theta\,\hat{\boldsymbol{\phi}}\int_r^R \bar{r}\,d\bar{r} \\
&= \left(\frac{\mu_0\omega\rho}{3}\right)\sin\theta\left[\frac{1}{r^2}\left(\frac{r^5}{5}\right) + \frac{r}{2}\left(R^2 - r^2\right)\right]\hat{\boldsymbol{\phi}} = \frac{\mu_0\omega\rho}{2}r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\hat{\boldsymbol{\phi}}. \\
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = \frac{\mu_0\omega\rho}{2}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right]\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left[r^2\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right]\hat{\boldsymbol{\theta}}\right\} \\
&= \mu_0\omega\rho\left[\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\cos\theta\,\hat{\mathbf{r}} - \left(\frac{R^2}{3} - \frac{2r^2}{5}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right]. \text{ But } \rho = \frac{Q}{(4/3)\pi R^3}, \text{ so} \\
&= \boxed{\frac{\mu_0\omega Q}{4\pi R}\left[\left(1 - \frac{3r^2}{5R^2}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1 - \frac{6r^2}{5R^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right].}
\end{aligned}$$

Problem 5.30

(a)
$$\left\{\begin{aligned}
-\frac{\partial W_z}{\partial x} &= F_y \Rightarrow W_z(x,y,z) = -\textstyle\int_0^x F_y(x',y,z)\,dx' + C_1(y,z). \\
\frac{\partial W_y}{\partial x} &= F_z \Rightarrow W_y(x,y,z) = +\textstyle\int_0^x F_z(x',y,z)\,dx' + C_2(y,z).
\end{aligned}\right\}$$

These satisfy (ii) and (iii), for *any* C_1 and C_2; it remains to choose these functions so as to satisfy (i):

$-\int_0^x \frac{\partial F_y(x',y,z)}{\partial y}\,dx' + \frac{\partial C_1}{\partial y} - \int_0^x \frac{\partial F_z(x',y,z)}{\partial z}\,dx' - \frac{\partial C_2}{\partial z} = F_x(x,y,z)$. But $\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$, so $\int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}\,dx' + \frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial z} = F_x(x,y,z)$. Now $\int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}\,dx' = F_x(x,y,z) - F_x(0,y,z)$, so $\frac{\partial C_1}{\partial y} - \frac{\partial C_2}{\partial z} = F_x(0,y,z)$. We may as well pick $C_2 = 0$, $C_1(y,z) = \int_0^y F_x(0,y',z)\,dy'$, and we're done, with

$$W_x = 0; \quad W_y = \int_0^x F_z(x',y,z)\,dx'; \quad W_z = \int_0^y F_x(0,y',z)\,dy' - \int_0^x F_y(x',y,z)\,dx'.$$

(b) $\boldsymbol{\nabla} \times \mathbf{W} = \left(\frac{\partial W_z}{\partial y} - \frac{\partial W_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial W_x}{\partial z} - \frac{\partial W_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial W_y}{\partial x} - \frac{\partial W_x}{\partial y}\right)\hat{\mathbf{z}}$

$= \left[F_x(0,y,z) - \int_0^x \frac{\partial F_y(x',y,z)}{\partial y}\,dx' - \int_0^x \frac{\partial F_z(x',y,z)}{\partial z}\,dx'\right]\hat{\mathbf{x}} + [0 + F_y(x,y,z)]\,\hat{\mathbf{y}} + [F_z(x,y,z) - 0]\,\hat{\mathbf{z}}$.

But $\boldsymbol{\nabla}\cdot\mathbf{F} = 0$, so the $\hat{\mathbf{x}}$ term is $\left[F_x(0,y,z) + \int_0^x \frac{\partial F_x(x',y,z)}{\partial x'}\,dx'\right] = F_x(0,y,z) + F_x(x,y,z) - F_x(0,y,z)$, so $\boldsymbol{\nabla}\times\mathbf{W} = \mathbf{F}$. ✓

$\boldsymbol{\nabla}\cdot\mathbf{W} = \frac{\partial W_x}{\partial x} + \frac{\partial W_y}{\partial y} + \frac{\partial W_z}{\partial z} = 0 + \int_0^x \frac{\partial F_z(x',y,z)}{\partial y}\,dx' + \int_0^y \frac{\partial F_x(0,y',z)}{\partial z}\,dy' - \int_0^x \frac{\partial F_y(x',y,z)}{\partial z}\,dx' \neq 0$, in general.

(c) $W_y = \int_0^x x'\,dx' = \frac{x^2}{2}$; $W_z = \int_0^y y'\,dy' - \int_0^x z\,dx' = \frac{y^2}{2} - zx$.

$$\boxed{\mathbf{W} = \frac{x^2}{2}\,\hat{\mathbf{y}} + \left(\frac{y^2}{2} - zx\right)\hat{\mathbf{z}}.} \qquad \boldsymbol{\nabla}\times\mathbf{W} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & x^2/2 & (y^2/2 - zx) \end{vmatrix} = y\,\hat{\mathbf{x}} + z\,\hat{\mathbf{y}} + x\,\hat{\mathbf{z}} = \mathbf{F}. \ ✓$$

Problem 5.31

(a) At the surface of the solenoid, $\mathbf{B}_{\text{above}} = 0$, $\mathbf{B}_{\text{below}} = \mu_0 n I\,\hat{\mathbf{z}} = \mu_0 K\,\hat{\mathbf{z}}$; $\hat{\mathbf{n}} = \hat{\mathbf{s}}$; so $\mathbf{K}\times\hat{\mathbf{n}} = -K\,\hat{\mathbf{z}}$. Evidently Eq. 5.74 holds. ✓

(b) In Eq. 5.67, both expressions reduce to $(\mu_0 R^2\omega\sigma/3)\sin\theta\,\hat{\boldsymbol{\phi}}$ at the surface, so Eq. 5.75 is satisfied. $\left.\frac{\partial\mathbf{A}}{\partial r}\right|_{R+} = \frac{\mu_0 R^4\omega\sigma}{3}\left(-\frac{2\sin\theta}{r^3}\right)\hat{\boldsymbol{\phi}}\bigg|_R = -\frac{2\mu_0 R\omega\sigma}{3}\sin\theta\,\hat{\boldsymbol{\phi}}$; $\left.\frac{\partial\mathbf{A}}{\partial r}\right|_{R-} = \frac{\mu_0 R\omega\sigma}{3}\sin\theta\,\hat{\boldsymbol{\phi}}$. So the left side of Eq. 5.76 is $-\mu_0 R\omega\sigma\sin\theta\,\hat{\boldsymbol{\phi}}$. Meanwhile $\mathbf{K} = \sigma\mathbf{v} = \sigma(\boldsymbol{\omega}\times\mathbf{r}) = \sigma\omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, so the right side of Eq. 5.76 is $-\mu_0\sigma\omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, and the equation is satisfied.

Problem 5.32

Because $\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}}$ at every point on the surface, it follows that $\frac{\partial\mathbf{A}}{\partial x}$ and $\frac{\partial\mathbf{A}}{\partial y}$ are the same above and below; any discontinuity is confined to the normal derivative.

$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \left(-\frac{\partial A_{y_{\text{above}}}}{\partial z} + \frac{\partial A_{y_{\text{below}}}}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_{x_{\text{above}}}}{\partial z} - \frac{\partial A_{x_{\text{below}}}}{\partial z}\right)\hat{\mathbf{y}}$. But Eq. 5.74 says this equals $\mu_0 K(-\hat{\mathbf{y}})$. So $\frac{\partial A_{y_{\text{above}}}}{\partial z} = \frac{\partial A_{y_{\text{below}}}}{\partial z}$, and $\frac{\partial A_{x_{\text{above}}}}{\partial z} - \frac{\partial A_{x_{\text{below}}}}{\partial z} = -\mu_0 K$. Thus the *normal* derivative of the *component of* $\mathbf{A}$ *parallel to* $\mathbf{K}$ suffers a discontinuity $-\mu_0 K$, or, more compactly: $\boxed{\frac{\partial\mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial\mathbf{A}_{\text{below}}}{\partial n} = -\mu_0\mathbf{K}.}$

Problem 5.33

(Same idea as Prob. 3.33.) Write $\mathbf{m} = (\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} + (\mathbf{m}\cdot\hat{\boldsymbol{\theta}})\,\hat{\boldsymbol{\theta}} = m\cos\theta\,\hat{\mathbf{r}} - m\sin\theta\,\hat{\boldsymbol{\theta}}$ (Fig. 5.54). Then $3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m} = 3m\cos\theta\,\hat{\mathbf{r}} - m\cos\theta\,\hat{\mathbf{r}} + m\sin\theta\,\hat{\boldsymbol{\theta}} = 2m\cos\theta\,\hat{\mathbf{r}} + m\sin\theta\,\hat{\boldsymbol{\theta}}$, and Eq. 5.87 ⇔ Eq. 5.86. qed

Problem 5.34

(a) $\mathbf{m} = I\mathbf{a} = \boxed{I\pi R^2\,\hat{\mathbf{z}}.}$

(b) $\mathbf{B} \approx \boxed{\dfrac{\mu_0}{4\pi}\dfrac{I\pi R^2}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right).}$

(c) On the z axis, $\theta = 0$, $r = z$, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ (for $z > 0$), so $\boxed{\mathbf{B} \approx \dfrac{\mu_0 I R^2}{2z^3}\,\hat{\mathbf{z}}}$ (for $z < 0$, $\theta = \pi$, $\hat{\mathbf{r}} = -\hat{\mathbf{z}}$, so the field is the same, with $|z|^3$ in place of z^3). The exact answer (Eq. 5.38) reduces (for $z \gg R$) to $B \approx \mu_0 I R^2/2|z|^3$, so they agree.

Problem 5.35

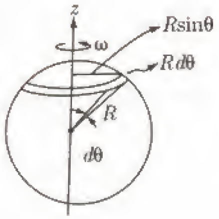
For a ring, $m = I\pi r^2$. Here $I \to \sigma v\,dr = \sigma\omega r\,dr$, so $m = \int_0^R \pi r^2 \sigma\omega r\,dr = \boxed{\pi\sigma\omega R^4/4.}$

Problem 5.36

The total charge on the shaded ring is $dq = \sigma(2\pi R\sin\theta)R\,d\theta$. The time for one revolution is $dt = 2\pi/\omega$. So the current in the ring is $I = \dfrac{dq}{dt} = \sigma\omega R^2\sin\theta\,d\theta$. The area of the ring is $\pi(R\sin\theta)^2$, so the magnetic moment of the ring is $dm = (\sigma\omega R^2\sin\theta\,d\theta)\pi R^2\sin^2\theta$, and the total dipole moment of the shell is

$m = \sigma\omega\pi R^4\int_0^\pi \sin^3\theta\,d\theta = (4/3)\sigma\omega\pi R^4$, or $\boxed{\mathbf{m} = \dfrac{4\pi}{3}\sigma\omega R^4\,\hat{\mathbf{z}}.}$

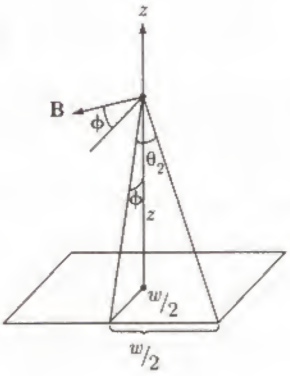
The dipole term in the multipole expansion for $\mathbf{A}$ is therefore $\mathbf{A}_{\text{dip}} = \dfrac{\mu_0}{4\pi}\dfrac{4\pi}{3}\sigma\omega R^4\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}} = \dfrac{\mu_0\sigma\omega R^4}{3}\dfrac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}$, which is also the *exact* potential (Eq. 5.67); evidently a spinning sphere produces a perfect dipole field, with no higher multipole contributions.



Problem 5.37

The field of one side is given by Eq. 5.35, with $s \to \sqrt{z^2 + (w/2)^2}$ and $\sin\theta_2 = -\sin\theta_1 = \dfrac{(w/2)}{\sqrt{z^2 + w^2/2}}$;

$B = \dfrac{\mu_0 I}{4\pi}\dfrac{w}{\sqrt{z^2 + (w^2/4)}\sqrt{z^2 + (w^2/2)}}$. To pick off the vertical component, multiply by $\sin\phi = \dfrac{(w/2)}{\sqrt{z^2 + (w/2)^2}}$; for all four sides, multiply by 4: $\boxed{\mathbf{B} = \dfrac{\mu_0 I}{2\pi}\dfrac{w^2}{(z^2 + w^2/4)\sqrt{z^2 + w^2/2}}\,\hat{\mathbf{z}}.}$ For $z \gg w$, $\mathbf{B} \approx \dfrac{\mu_0 I w^2}{2\pi z^3}\,\hat{\mathbf{z}}$. The field of a dipole $\boxed{m = Iw^2,}$ for points on the z axis (Eq. 5.86, with $r \to z$, $\hat{\mathbf{r}} \to \hat{\mathbf{z}}$, $\theta = 0$) is $\mathbf{B} = \dfrac{\mu_0}{2\pi}\dfrac{m}{z^3}\,\hat{\mathbf{z}}$. ✓



Problem 5.38

The mobile charges *do* pull in toward the axis, but the resulting concentration of (negative) charge sets up an *electric* field that repels away further accumulation. Equilibrium is reached when the electric repulsion on a mobile charge q balances the magnetic attraction: $\mathbf{F} = q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})] = 0 \Rightarrow \mathbf{E} = -(\mathbf{v} \times \mathbf{B})$. Say the current is in the z direction: $\mathbf{J} = \rho_- v\,\hat{\mathbf{z}}$ (where ρ_- and v are both negative).

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi s = \mu_0 J \pi s^2 \Rightarrow \mathbf{B} = \frac{\mu_0 \rho_- v s}{2}\,\hat{\boldsymbol{\phi}};$$

$$\int \mathbf{E}\cdot d\mathbf{a} = E\,2\pi s l = \frac{1}{\epsilon_0}(\rho_+ + \rho_-)\pi s^2 l \Rightarrow \mathbf{E} = \frac{1}{2\epsilon_0}(\rho_+ + \rho_-)s\,\hat{\mathbf{s}}.$$

$$\frac{1}{2\epsilon_0}(\rho_+ + \rho_-)s\,\hat{\mathbf{s}} = -\left[(v\,\hat{\mathbf{z}})\times\left(\frac{\mu_0\rho_- v s}{2}\,\hat{\boldsymbol{\phi}}\right)\right] = \frac{\mu_0}{2}\rho_- v^2 s\,\hat{\mathbf{s}} \Rightarrow \rho_+ + \rho_- = \rho_-(\epsilon_0\mu_0 v^2) = \rho_-\left(\frac{v^2}{c^2}\right).$$

Evidently $\rho_+ = -\rho_-\left(1 - \frac{v^2}{c^2}\right) = \frac{\rho_-}{\gamma^2}$, or $\rho_- = -\gamma^2\rho_+$. In this naive model, the mobile negative charges fill a smaller inner cylinder, leaving a shell of positive (stationary) charge at the outside. But since $v \ll c$, the effect is extremely small.

Problem 5.39

(a) If *positive* charges flow to the *right*, they are deflected $\boxed{\text{down,}}$ and the bottom plate acquires a *positive* charge.

(b) $qvB = qE \Rightarrow E = vB \Rightarrow V = Et = \boxed{vBt,}$ with the *bottom* at higher potential.

(c) If *negative* charges flow to the *left*, they are *also* deflected down, and the bottom plate acquires a *negative* charge. The potential difference is still the same, but this time the *top* plate is at the higher potential.

Problem 5.40

From Eq. 5.17, $\mathbf{F} = I\int(d\mathbf{l}\times\mathbf{B})$. But $\mathbf{B}$ is constant, in this case, so it comes outside the integral: $\mathbf{F} = I\left(\int d\mathbf{l}\right)\times\mathbf{B}$, and $\int d\mathbf{l} = \mathbf{w}$, the vector displacement from the point at which the wire first enters the field to the point where it leaves. Since $\mathbf{w}$ and $\mathbf{B}$ are perpendicular, $F = IBw$, and $\mathbf{F}$ is perpendicular to $\mathbf{w}$.

Problem 5.41

The angular momentum acquired by the particle as it moves out from the center to the edge is

$$\mathbf{L} = \int \frac{d\mathbf{L}}{dt}\,dt = \int \mathbf{N}\,dt = \int(\mathbf{r}\times\mathbf{F})\,dt = \int \mathbf{r}\times q(\mathbf{v}\times\mathbf{B})\,dt = q\int \mathbf{r}\times(d\mathbf{l}\times\mathbf{B}) = q\left[\int(\mathbf{r}\cdot\mathbf{B})\,d\mathbf{l} - \int \mathbf{B}(\mathbf{r}\cdot d\mathbf{l})\right].$$

But $\mathbf{r}$ is perpendicular to $\mathbf{B}$, so $\mathbf{r}\cdot\mathbf{B} = 0$, and $\mathbf{r}\cdot d\mathbf{l} = \mathbf{r}\cdot d\mathbf{r} = \frac{1}{2}d(\mathbf{r}\cdot\mathbf{r}) = \frac{1}{2}d(r^2) = r\,dr = (1/2\pi)(2\pi r\,dr)$. So $\mathbf{L} = -\frac{q}{2\pi}\int_0^R \mathbf{B}\,2\pi r\,dr = -\frac{q}{2\pi}\int \mathbf{B}\,da$. It follows that $\boxed{L = -\frac{q}{2\pi}\Phi,}$ where $\Phi = \int B\,da$ is the total flux. In particular, if $\Phi = 0$, then $L = 0$, and the charge emerges with zero angular momentum, which means it is going along a radial line. qed

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Problem 5.42

From Eq. 5.24, $\mathbf{F} = \int(\mathbf{K}\times\mathbf{B}_{\text{ave}})\,da$. Here $\mathbf{K} = \sigma\mathbf{v}$, $\mathbf{v} = \omega R\sin\theta\,\hat{\boldsymbol{\phi}}$, $da = R^2\sin\theta\,d\theta\,d\phi$, and $\mathbf{B}_{\text{ave}} = \frac{1}{2}(\mathbf{B}_{\text{in}} + \mathbf{B}_{\text{out}})$. From Eq. 5.68,

$$\begin{aligned}
\mathbf{B}_{\text{in}} &= \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\omega(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}).\text{ From Eq. 5.67,}\\
\mathbf{B}_{\text{out}} &= \boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\times\left(\frac{\mu_0R^4\omega\sigma}{3}\frac{\sin\theta}{r^2}\,\hat{\boldsymbol{\phi}}\right) = \frac{\mu_0R^4\omega\sigma}{3}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sin^2\theta}{r^2}\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}}\right]\\
&= \frac{\mu_0R^4\omega\sigma}{3r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) = \frac{\mu_0R\omega\sigma}{3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\text{ (since } r = R).\\
\mathbf{B}_{\text{ave}} &= \frac{\mu_0R\omega\sigma}{6}(4\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}).\\
\mathbf{K}\times\mathbf{B}_{\text{ave}} &= (\sigma\omega R\sin\theta)\left(\frac{\mu_0R\omega\sigma}{6}\right)\left[\hat{\boldsymbol{\phi}}\times(4\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})\right] = \frac{\mu_0}{6}(\sigma\omega R)^2(4\cos\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\hat{\mathbf{r}})\sin\theta.
\end{aligned}$$

Picking out the z component of $\hat{\boldsymbol{\theta}}$ (namely, $-\sin\theta$) and of $\hat{\mathbf{r}}$ (namely, $\cos\theta$), we have $(\mathbf{K}\times\mathbf{B}_{\text{ave}})_z = -\frac{\mu_0}{2}(\sigma\omega R)^2\sin^2\theta\cos\theta$, so

$$F_z = -\frac{\mu_0}{2}(\sigma\omega R)^2R^2\int\sin^3\theta\cos\theta\,d\theta\,d\phi = -\frac{\mu_0}{2}(\sigma\omega R^2)^2\,2\pi\left(\frac{\sin^4\theta}{4}\right)\Bigg|_0^{\pi/2},\text{ or }\boxed{\mathbf{F} = -\frac{\mu_0\pi}{4}(\sigma\omega R^2)^2\,\hat{\mathbf{z}}.}$$

Problem 5.43

(a) $\mathbf{F} = m\mathbf{a} = q_e(\mathbf{v}\times\mathbf{B}) = \frac{\mu_0}{4\pi}\frac{q_eq_m}{r^2}(\mathbf{v}\times\hat{\mathbf{r}})$; $\boxed{\mathbf{a} = \frac{\mu_0}{4\pi}\frac{q_eq_m}{mr^3}(\mathbf{v}\times\mathbf{r}).}$

(b) Because $\mathbf{a}\perp\mathbf{v}$, $\mathbf{a}\cdot\mathbf{v} = 0$. But $\mathbf{a}\cdot\mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = \frac{1}{2}\frac{d}{dt}(v^2) = v\frac{dv}{dt}$. So $\frac{dv}{dt} = 0$. qed

(c) $\frac{d\mathbf{Q}}{dt} = m(\mathbf{v}\times\mathbf{v}) + m(\mathbf{r}\times\mathbf{a}) - \frac{\mu_0q_eq_m}{4\pi}\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = 0 + \frac{\mu_0q_eq_m}{4\pi r^3}[\mathbf{r}\times(\mathbf{v}\times\mathbf{r}] - \frac{\mu_0q_eq_m}{4\pi}\left(\frac{\mathbf{v}}{r} - \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right)$
$= \frac{\mu_0q_eq_m}{4\pi}\left\{\frac{1}{r^3}[r^2\mathbf{v} - (\mathbf{r}\cdot\mathbf{v})\mathbf{r}] - \frac{\mathbf{v}}{r} + \frac{\mathbf{r}}{r^2}\frac{d}{dt}(\sqrt{\mathbf{r}\cdot\mathbf{r}})\right\} = \frac{\mu_0q_eq_m}{4\pi}\left[\frac{\mathbf{v}}{r} - \frac{(\hat{\mathbf{r}}\cdot\mathbf{v})}{r}\hat{\mathbf{r}} - \frac{\mathbf{v}}{r} + \frac{\hat{\mathbf{r}}}{2r}\frac{2(\mathbf{r}\cdot\mathbf{v})}{r}\right] = 0.$ ✓

(d) (i) $\mathbf{Q}\cdot\hat{\boldsymbol{\phi}} = Q(\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\phi}}) = m(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\phi}} - \frac{\mu_0q_eq_m}{4\pi}(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}})$. But $\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = 0$, so $(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\phi}} = 0$. But $\mathbf{r} = r\,\hat{\mathbf{r}}$, and $\mathbf{v} = \frac{d\mathbf{l}}{dt} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\,\hat{\boldsymbol{\phi}}$ (where dots denote differentiation with respect to time), so

$$\mathbf{r}\times\mathbf{v} = \begin{vmatrix}\hat{\mathbf{r}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}}\\ r & 0 & 0\\ \dot{r} & r\dot{\theta} & r\sin\theta\dot{\phi}\end{vmatrix} = (-r^2\sin\theta\dot{\phi})\,\hat{\boldsymbol{\theta}} + (r^2\dot{\theta})\,\hat{\boldsymbol{\phi}}.$$

Therefore $(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\phi}} = r^2\dot{\theta} = 0$, so θ is constant. qed

(ii) $\mathbf{Q}\cdot\hat{\mathbf{r}} = Q(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}) = m(\mathbf{r}\times\mathbf{v})\cdot\hat{\mathbf{r}} - \frac{\mu_0q_eq_m}{4\pi}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})$. But $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = \cos\theta$, and $(\mathbf{r}\times\mathbf{v})\perp\mathbf{r}\Rightarrow(\mathbf{r}\times\mathbf{v})\cdot\hat{\mathbf{r}} = 0$, so $Q\cos\theta = -\frac{\mu_0q_eq_m}{4\pi}$, or $Q = -\frac{\mu_0q_eq_m}{4\pi\cos\theta}$. And since θ is constant, so too is Q. qed

(iii) $\mathbf{Q}\cdot\hat{\boldsymbol{\theta}} = Q(\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\theta}}) = m(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\theta}} - \frac{\mu_0q_eq_m}{4\pi}(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}})$. But $\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\theta}} = -\sin\theta$, $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}} = 0$, and $(\mathbf{r}\times\mathbf{v})\cdot\hat{\boldsymbol{\theta}} = -r^2\sin\theta\dot{\phi}$ (from (i)), so $-Q\sin\theta = -mr^2\sin\theta\dot{\phi}\Rightarrow\dot{\phi} = \frac{Q}{mr^2} = \frac{k}{r^2}$, with $\boxed{k \equiv \frac{Q}{m} = -\frac{\mu_0q_eq_m}{4\pi m\cos\theta}.}$

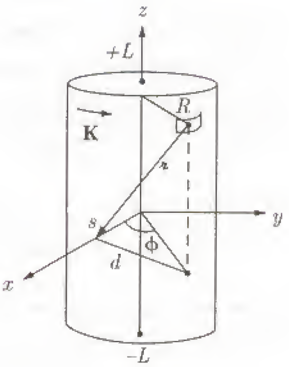
(e) $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$, but $\dot{\theta} = 0$ and $\dot{\phi} = \frac{k}{r^2}$, so $\dot{r}^2 = v^2 - r^2\sin^2\theta\frac{k^2}{r^4} = v^2 - \frac{k^2\sin^2\theta}{r^2}$.

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{v^2-(k\sin\theta/r)^2}{(k^2/r^4)} = r^2\left[\left(\frac{vr}{k}\right)^2-\sin^2\theta\right];\ \boxed{\frac{dr}{d\phi} = r\sqrt{\left(\frac{vr}{k}\right)^2-\sin^2\theta}.}$$

(f) $\displaystyle\int\frac{dr}{r\sqrt{(vr/k)^2-\sin^2\theta}} = \int d\phi \Rightarrow \phi-\phi_0 = \frac{1}{\sin\theta}\sec^{-1}\left(\frac{vr}{k\sin\theta}\right);\ \sec[(\phi-\phi_0)\sin\theta] = \frac{vr}{k\sin\theta}$, or

$\boxed{r(\phi) = \dfrac{A}{\cos[(\phi-\phi_0)\sin\theta]},}$ where $A \equiv -\dfrac{\mu_0 q_e q_m\tan\theta}{4\pi m v}$.

Problem 5.44



Put the field point on the x axis, so $\mathbf{r} = (s,0,0)$. Then $\mathbf{B} = \dfrac{\mu_0}{4\pi}\displaystyle\int\frac{(\mathbf{K}\times\hat{\mathscr{r}})}{\mathscr{r}^2}\,da$; $da = R\,d\phi\,dz$; $\mathbf{K} = K\hat{\boldsymbol{\phi}} = K(-\sin\phi\,\hat{\mathbf{x}}+\cos\phi\,\hat{\mathbf{y}})$; $\boldsymbol{\mathscr{r}} = (s-R\cos\phi)\,\hat{\mathbf{x}} - R\sin\phi\,\hat{\mathbf{y}} - z\,\hat{\mathbf{z}}$.

$$\mathbf{K}\times\boldsymbol{\mathscr{r}} = K\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ -\sin\phi & \cos\phi & 0\\ (s-R\cos\phi) & (-R\sin\phi) & (-z)\end{vmatrix} =$$

$K\left[(-z\cos\phi)\,\hat{\mathbf{x}} + (-z\sin\phi)\,\hat{\mathbf{y}} + (R-s\cos\phi)\,\hat{\mathbf{z}}\right]$; $\mathscr{r}^2 = z^2+R^2+s^2-2Rs\cos\phi$. The x and y components integrate to zero (z integrand is odd, as in Prob. 5.17).

$$\begin{aligned}B_z &= \frac{\mu_0}{4\pi}KR\int\frac{(R-s\cos\phi)}{(z^2+R^2+s^2-2Rs\cos\phi)^{3/2}}\,d\phi\,dz\\ &= \frac{\mu_0 KR}{4\pi}\int_0^{2\pi}(R-s\cos\phi)\left\{\int_{-\infty}^{\infty}\frac{dz}{(z^2+d^2)^{3/2}}\right\}d\phi,\end{aligned}$$

where $d^2 \equiv R^2+s^2-2Rs\cos\phi$. Now $\displaystyle\int_{-\infty}^{\infty}\frac{dz}{(z^2+d^2)^{3/2}} = \left.\frac{2z}{d^2\sqrt{z^2+d^2}}\right|_0^{\infty} = \frac{2}{d^2}$.

$$\begin{aligned}&= \frac{\mu_0 KR}{2\pi}\int_0^{2\pi}\frac{(R-s\cos\phi)}{(R^2+s^2-2Rs\cos\phi)}\,d\phi;\ (R-s\cos\phi) = \frac{1}{2R}\left[(R^2-s^2)+(R^2+s^2-2Rs\cos\phi)\right].\\ &= \frac{\mu_0 K}{4\pi}\left[(R^2-s^2)\int_0^{2\pi}\frac{d\phi}{(R^2+s^2-2Rs\cos\phi)}+\int_0^{2\pi}d\phi\right].\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi}{a+b\cos\phi} &= 2\int_0^{\pi}\frac{d\phi}{a+b\cos\phi} = \frac{4}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(\phi/2)}{a+b}\right]\Bigg|_0^{\pi}\\ &= \frac{4}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(\pi/2)}{a+b}\right] = \frac{4}{\sqrt{a^2-b^2}}\left(\frac{\pi}{2}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}.\end{aligned}$$

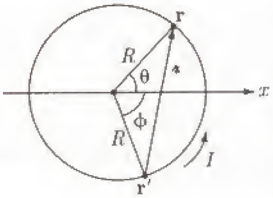
Here $a = R^2+s^2$, $b = -2Rs$, so $a^2-b^2 = R^4+2R^2s^2+s^4-4R^2s^2 = R^4-2R^2s^2+s^4 = (R^2-s^2)^2$; $\sqrt{a^2-b^2} = |R^2-d^2|$.

$$B_z = \frac{\mu_0 K}{4\pi}\left[\frac{(R^2-s^2)}{|R^2-s^2|}2\pi+2\pi\right] = \frac{\mu_0 K}{2}\left(\frac{R^2-s^2}{|R^2-s^2|}+1\right).$$

Inside the solenoid, $s<R$, so $B_z = \dfrac{\mu_0 K}{2}(1+1) = \mu_0 K$. *Outside* the solenoid, $s>R$, so $B_z = \dfrac{\mu_0 K}{2}(-1+1) = 0$. Here $K = nI$, so $\boxed{\mathbf{B} = \mu_0 nI\,\hat{\mathbf{z}}\text{(inside), and 0(outside)}}$ (as we found more easily using Ampére's law, in Ex. 5.9).

Problem 5.45

Let the source point be $\mathbf{r}' = R\cos\phi\,\hat{\mathbf{x}} - R\sin\phi\,\hat{\mathbf{y}}$, and the field point be $\mathbf{r} = R\cos\theta\,\hat{\mathbf{x}} + R\sin\theta\,\hat{\mathbf{y}}$; then $\boldsymbol{\mathscr{r}} = R\left[(\cos\theta - \cos\phi)\,\hat{\mathbf{x}} + (\sin\theta + \sin\phi)\,\hat{\mathbf{y}}\right]$ and $d\mathbf{l} = R\sin\phi\,d\phi\,\hat{\mathbf{x}} + R\cos\phi\,d\phi\,\hat{\mathbf{y}} = R\,d\phi(\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})$.



$$\begin{aligned}
d\mathbf{l}\times\boldsymbol{\mathscr{r}} &= R^2\,d\phi\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \sin\phi & \cos\phi & 0 \\ (\cos\theta - \cos\phi) & (\sin\theta + \sin\phi) & 0 \end{vmatrix} \\
&= R^2(\sin\phi\sin\theta + \sin^2\phi - \cos\theta\cos\phi + \cos^2\phi)\,d\phi\,\hat{\mathbf{z}} \\
&= R^2(1 + \sin\theta\sin\phi - \cos\theta\cos\phi)\,d\phi\,\hat{\mathbf{z}} = R^2\left[1 - \cos(\theta+\phi)\right]\,d\phi\,\hat{\mathbf{z}}. \\
\mathbf{B} &= \frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{l}\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi}R^2\,\hat{\mathbf{z}}\int_0^\pi \frac{\left[1-\cos(\theta+\phi)\right]}{\left[2R^2 - 2R^2\cos(\theta+\phi)\right]^{3/2}}\,d\phi = \frac{\mu_0 I R^2}{4\pi(2R^2)^{3/2}}\,\hat{\mathbf{z}}\int_0^\pi\frac{d\phi}{\sqrt{1-\cos(\theta+\phi)}} \\
&= \frac{\mu_0 I}{8\sqrt{2}\pi R}\,\hat{\mathbf{z}}\int_0^\pi\frac{d\phi}{\sqrt{2}\sin[(\theta+\phi)/2]} = \frac{\mu_0 I}{16\pi R}\,\hat{\mathbf{z}}\left\{2\ln\left[\tan\left(\frac{\theta+\phi}{4}\right)\right]\right\}\Bigg|_0^\pi = \boxed{\frac{\mu_0 I}{8\pi R}\ln\left[\frac{\tan\left(\frac{\theta+\pi}{4}\right)}{\tan\left(\frac{\theta}{4}\right)}\right]\,\hat{\mathbf{z}}.}
\end{aligned}$$

Problem 5.46

(a) From Eq. 5.38, $\boxed{\mathbf{B} = \frac{\mu_0 I R^2}{2}\left\{\frac{1}{\left[R^2 + (d/2+z)^2\right]^{3/2}} + \frac{1}{\left[R^2 + (d/2-z)^2\right]^{3/2}}\right\}.}$

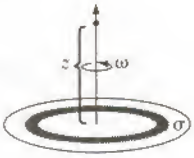
$$\begin{aligned}
\frac{\partial B}{\partial z} &= \frac{\mu_0 I R^2}{2}\left\{\frac{(-3/2)2(d/2+z)}{\left[R^2 + (d/2+z)^2\right]^{5/2}} + \frac{(-3/2)2(d/2-z)(-1)}{\left[R^2 + (d/2-z)^2\right]^{5/2}}\right\} \\
&= \frac{3\mu_0 I R^2}{2}\left\{\frac{-(d/2+z)}{\left[R^2 + (d/2+z)^2\right]^{5/2}} + \frac{(d/2-z)}{\left[R^2 + (d/2-z)^2\right]^{5/2}}\right\}. \\
\left.\frac{\partial B}{\partial z}\right|_{z=0} &= \frac{3\mu_0 I R^2}{2}\left\{\frac{-d/2}{\left[R^2 + (d/2)^2\right]^{5/2}} + \frac{d/2}{\left[R^2 + (d/2)^2\right]^{5/2}}\right\} = 0.\ \checkmark
\end{aligned}$$

(b) Differentiating again:

$$\begin{aligned}
\frac{\partial^2 B}{\partial z^2} &= \frac{3\mu_0 I R^2}{2}\left\{\frac{-1}{\left[R^2 + (d/2+z)^2\right]^{5/2}} + \frac{-(d/2+z)(-5/2)2(d/2+z)}{\left[R^2 + (d/2+z)^2\right]^{7/2}}\right. \\
&\quad \left. + \frac{-1}{\left[R^2 + (d/2-z)^2\right]^{5/2}} + \frac{(d/2-z)(-5/2)2(d/2-z)(-1)}{\left[R^2 + (d/2-z)^2\right]^{7/2}}\right\}. \\
\left.\frac{\partial^2 B}{\partial z^2}\right|_{z=0} &= \frac{3\mu_0 I R^2}{2}\left\{\frac{-2}{\left[R^2 + (d/2)^2\right]^{5/2}} + \frac{2(5/2)2(d/2)^2 2}{\left[R^2 + (d/2)^2\right]^{7/2}}\right\} = \frac{3\mu_0 I R^2}{\left[R^2 + (d/2)^2\right]^{7/2}}\left(-R^2 - \frac{d^2}{4} + \frac{5d^2}{4}\right) \\
&= \frac{3\mu_0 I R^2}{\left[R^2 + (d/2)^2\right]^{7/2}}\left(d^2 - R^2\right).\quad \text{Zero if } \boxed{d = R,} \text{ in which case} \\
B(0) &= \frac{\mu_0 I R^2}{2}\left\{\frac{1}{\left[R^2 + (R/2)^2\right]^{3/2}} + \frac{1}{\left[R^2 + (R/2)^2\right]^{3/2}}\right\} = \mu_0 I R^2\frac{1}{(5R^2/4)^{3/2}} = \boxed{\frac{8\mu_0 I}{5^{3/2}R}.}
\end{aligned}$$

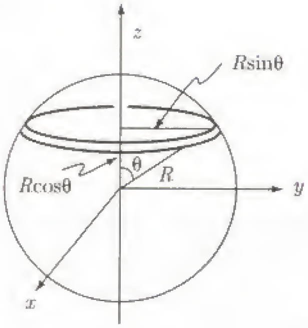
Problem 5.47

(a) The total charge on the shaded ring is $dq = \sigma(2\pi r)\,dr$. The time for one revolution is $dt = 2\pi/\omega$. So the current in the ring is $I = \dfrac{dq}{dt} = \sigma\omega r\,dr$. From Eq. 5.38, the magnetic field of this ring (for points on the axis) is $d\mathbf{B} = \dfrac{\mu_0}{2}\sigma\omega r\dfrac{r^2}{(r^2+z^2)^{3/2}}\,dr\,\hat{\mathbf{z}}$, and the total field of the disk is



$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3\,dr}{(r^2+z^2)^{3/2}}\,\hat{\mathbf{z}}. \quad \text{Let } u \equiv r^2, \text{ so } du = 2r\,dr. \quad \text{Then} \\
&= \frac{\mu_0\sigma\omega}{4}\int_0^{R^2}\frac{u\,du}{(u+z^2)^{3/2}} = \frac{\mu_0\sigma\omega}{4}\left[2\left(\frac{u+2z^2}{\sqrt{u+z^2}}\right)\right]\Bigg|_0^{R^2} = \boxed{\frac{\mu_0\sigma\omega}{2}\left[\frac{(R^2+2z^2)}{\sqrt{R^2+z^2}}-2z\right]\hat{\mathbf{z}}.}
\end{aligned}$$

(b) Slice the sphere into slabs of thickness t, and use (a). Here $t = |d(R\cos\theta)| = R\sin\theta\,d\theta$; $\sigma \to \rho t = \rho R\sin\theta\,d\theta; R \to R\sin\theta;\ z \to z - R\cos\theta$. First rewrite the term in square brackets:



$$\begin{aligned}
\left[\frac{(R^2+2z^2)}{\sqrt{R^2+z^2}}-2z\right] &= \frac{2(R^2+z^2)}{\sqrt{R^2+z^2}} - \frac{R^2}{\sqrt{R^2+z^2}} - 2z \\
&= 2\left[\sqrt{R^2+z^2} - \frac{R^2/2}{\sqrt{R^2+z^2}} - z\right].
\end{aligned}$$

But $R^2+z^2 \to R^2\sin^2\theta + (z^2 - 2Rz\cos\theta + R^2\cos^2\theta) = R^2 + z^2 - 2Rz\cos\theta$. So

$$\begin{aligned}
B_z &= \frac{\mu_0\rho R\omega}{2}2\int_0^{\pi}\sin\theta\,d\theta\left[\sqrt{R^2+z^2-2Rz\cos\theta} - \frac{(R^2/2)\sin^2\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}} - (z - R\cos\theta)\right]. \\
&\quad \text{Let } u \equiv \cos\theta, \text{ so } du = -\sin\theta\,d\theta;\ \theta: 0 \to \pi \Rightarrow u: 1 \to -1;\ \sin^2\theta = 1-u^2. \\
&= \mu_0\rho R\omega\int_{-1}^{1}\left[\sqrt{R^2+z^2-2Rzu} - \frac{(R^2/2)(1-u^2)}{\sqrt{R^2+z^2-2Rzu}} - z + Ru\right]du \\
&= \mu_0\rho R\omega\left[I_1 - \frac{R^2}{2}(I_2 - I_3) - I_4 + I_5\right].
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-1}^{1}\sqrt{R^2+z^2-2Rzu}\,du = -\frac{1}{3Rz}\left(R^2+z^2-2Rzu\right)^{3/2}\Big|_{-1}^{1} \\
&= -\frac{1}{3Rz}\left[\left(R^2+z^2-2Rz\right)^{3/2} - \left(R^2+z^2+2Rz\right)^{3/2}\right] = -\frac{1}{3Rz}\left[(z-R)^3 - (z+R)^3\right] \\
&= -\frac{1}{3Rz}\left(z^3 - 3z^2R + 3zR^2 - R^3 - z^3 - 3z^2R - 3zR^2 - R^3\right) = \frac{2}{3z}(3z^2+R^2). \\
I_2 &= \int_{-1}^{1}\frac{1}{\sqrt{R^2+z^2-2Rzu}}\,du = -\frac{1}{Rz}\sqrt{R^2+z^2-2Rzu}\Big|_{-1}^{1} = -\frac{1}{Rz}\left[(z-R)-(z+R)\right] = \frac{2}{z}.
\end{aligned}$$

$$\begin{aligned}
I_3 &= \int_{-1}^{1} \frac{u^2}{\sqrt{R^2+z^2-2Rzu}}\,du \\
&= -\frac{1}{60R^3z^3}\left[8(R^2+z^2)^2+4(R^2+z^2)2Rzu+3(2Rz)^2u^2\right]\sqrt{R^2+z^2-2Rzu}\,\Big|_{-1}^{1} \\
&= -\frac{1}{60R^3z^3}\Big\{\left[8(R^2+z^2)^2+8Rz(R^2+z^2)+12R^2z^2\right](z-R) \\
&\quad - \left[8(R^2+z^2)^2-8Rz(R^2+z^2)+12R^2z^2\right](z+R)\Big\} \\
&= -\frac{1}{60R^3z^3}\left\{z\left[16Rz(R^2+z^2)\right]-R\left[16(R^2+z^2)^2+24R^2z^2\right]\right\} \\
&= -\frac{1}{60R^3z^3}16R\left(R^2z^2+z^4-R^4-2R^2z^2-z^4-\frac{3}{2}R^2z^2\right) \\
&= -\frac{4}{15R^2z^3}\left(-\frac{5}{2}R^2z^2-R^4\right)=\frac{4}{15z^3}\left(R^2+\frac{5}{2}z^2\right). \quad I_4 = z\int_{-1}^{1}du = 2z; \quad I_5 = R\int_{-1}^{1}u\,du = 0.
\end{aligned}$$

$$\begin{aligned}
B_z &= \mu_0 R\rho\omega\left[\frac{2}{3z}(3z^2+R^2)-\frac{R^2}{2}\frac{2}{z}+\frac{R^2}{2}\frac{4}{15z^3}\left(R^2+\frac{5}{2}z^2\right)-2z\right] \\
&= \mu_0 R\rho\omega\left(2z+\frac{2R^2}{3z}-\frac{R^2}{z}+\frac{2R^4}{15z^3}+\frac{R^2}{3z}-2z\right) \\
&= \mu_0\rho\omega\frac{2R^5}{15z^3}. \quad \text{But } \rho=\frac{Q}{(4/3)\pi R^3}, \text{ so } \boxed{\mathbf{B}=\frac{\mu_0 Q\omega R^2}{10\pi z^3}\,\hat{\mathbf{z}}.}
\end{aligned}$$

Problem 5.48

$\mathbf{B}=\dfrac{\mu_0 I}{4\pi}\displaystyle\int\frac{d\mathbf{l}'\times\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}$. $\boldsymbol{\mathscr{r}} = -R\cos\phi\,\hat{\mathbf{x}}+(y-R\sin\phi)\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}$. (For simplicity I'll drop the prime on ϕ.) $\mathscr{r}^2=R^2\cos^2\phi+y^2-2Ry\sin\phi+R^2\sin^2\phi+z^2=R^2+y^2+z^2-2Ry\sin\phi$. The source coordinates (x',y',z') satisfy $x'=R\cos\phi\Rightarrow dx'=-R\sin\phi\,d\phi$; $y'=R\sin\phi\Rightarrow dy'=R\cos\phi\,d\phi; z'=0\Rightarrow dz'=0$. So $d\mathbf{l}'=-R\sin\phi\,d\phi\,\hat{\mathbf{x}}+R\cos\phi\,d\phi\,\hat{\mathbf{y}}$.

$$d\mathbf{l}'\times\boldsymbol{\mathscr{r}}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ -R\sin\phi\,d\phi & R\cos\phi\,d\phi & 0\\ -R\cos\phi & (y-R\sin\phi) & z\end{vmatrix}=(Rz\cos\phi\,d\phi)\,\hat{\mathbf{x}}+(Rz\sin\phi\,d\phi)\,\hat{\mathbf{y}}+(-Ry\sin\phi\,d\phi+R^2\,d\phi)\,\hat{\mathbf{z}}.$$

$$B_x=\frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\cos\phi\,d\phi}{(R^2+y^2+z^2-2Ry\sin\phi)^{3/2}}=\frac{\mu_0 IRz}{4\pi}\frac{1}{Ry}\frac{1}{\sqrt{R^2+y^2+z^2-2Ry\sin\phi}}\Bigg|_0^{2\pi}=0,$$

since $\sin\phi=0$ at both limits. The y and z components are elliptic integrals, and cannot be expressed in terms of elementary functions.

$$\boxed{B_x=0;\quad B_y=\frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\sin\phi\,d\phi}{(R^2+y^2+z^2-2Ry\sin\phi)^{3/2}};\,B_z=\frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{(R-y\sin\phi)\,d\phi}{(R^2+y^2+z^2-2Ry\sin\phi)^{3/2}}.}$$

Problem 5.49

From the Biot-Savart law, the field of loop #1 is $\mathbf{B}=\dfrac{\mu_0 I_1}{4\pi}\displaystyle\oint_1\frac{d\mathbf{l}_1\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$; the force on loop #2 is

$$\mathbf{F}=I_2\oint_2 d\mathbf{l}_2\times\mathbf{B}=\frac{\mu_0}{4\pi}I_1I_2\oint_1\oint_2\frac{d\mathbf{l}_2\times(d\mathbf{l}_1\times\hat{\boldsymbol{\mathscr{r}}})}{\mathscr{r}^2}.$$ Now $d\mathbf{l}_2\times(d\mathbf{l}_1\times\hat{\boldsymbol{\mathscr{r}}})=d\mathbf{l}_1(d\mathbf{l}_2\cdot\hat{\boldsymbol{\mathscr{r}}})-\hat{\boldsymbol{\mathscr{r}}}(d\mathbf{l}_1\cdot d\mathbf{l}_2)$, so

$$\mathbf{F} = -\frac{\mu_0}{4\pi} I_1 I_2 \left\{ \oint\oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}(d\mathbf{l}_1 \cdot d\mathbf{l}_2) - \oint d\mathbf{l}_1 \oint \frac{(d\mathbf{l}_2 \cdot \hat{\boldsymbol{\mathscr{r}}})}{\mathscr{r}^2} \right\}$$

The first term is what we want. It remains to show that the second term is zero: $\boldsymbol{\mathscr{r}} = (x_2 - x_1)\,\hat{\mathbf{x}} + (y_2 - y_1)\,\hat{\mathbf{y}} + (z_2 - z_1)\,\hat{\mathbf{z}}$, so $\boldsymbol{\nabla}_2(1/\mathscr{r}) = \frac{\partial}{\partial x_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\hat{\mathbf{x}}$ $+ \frac{\partial}{\partial y_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\hat{\mathbf{y}} + \frac{\partial}{\partial z_2}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}\hat{\mathbf{z}}$ $= -\frac{(x_2 - x_1)}{\mathscr{r}^3}\,\hat{\mathbf{x}} - \frac{(y_2 - y_1)}{\mathscr{r}^3}\,\hat{\mathbf{y}} - \frac{(z_2 - z_1)}{\mathscr{r}^3}\,\hat{\mathbf{z}} = -\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3} = -\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$. So $\oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot d\mathbf{l}_2 = -\oint \boldsymbol{\nabla}_2\left(\frac{1}{\mathscr{r}}\right) \cdot d\mathbf{l}_2 = 0$ (by Corollary 2 in Sect. 1.3.3). qed

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Problem 5.50

Poisson's equation (Eq. 2.24) says $\nabla^2 V = -\frac{1}{\epsilon_0}\rho$. For *dielectrics* (with no *free* charge), $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$ (Eq. 4.12), and the resulting potential is $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\mathbf{P}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$. In *general*, $\rho = \epsilon_0\boldsymbol{\nabla}\cdot\mathbf{E}$ (Gauss's law), so the analogy is $\mathbf{P} \to -\epsilon_0\mathbf{E}$, and hence $V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{\mathbf{E}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$. qed

[There are many other ways to obtain this result. For example, using Eq. 1.100:

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = -\boldsymbol{\nabla}'\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\boldsymbol{\mathscr{r}}) = 4\pi\delta^3(\mathbf{r} - \mathbf{r}'),$$

$$V(\mathbf{r}) = \int V(\mathbf{r}')\delta^3(\mathbf{r} - \mathbf{r}')\,d\tau' = -\frac{1}{4\pi}\int V(\mathbf{r}')\boldsymbol{\nabla}'\cdot\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)d\tau' = \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot\left[\boldsymbol{\nabla}'V(\mathbf{r}')\right]d\tau' - \frac{1}{4\pi}\oint V(\mathbf{r}')\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\cdot d\mathbf{a}'$$

(Eq. 1.59). But $\boldsymbol{\nabla}'V(\mathbf{r}') = -\mathbf{E}(\mathbf{r}')$, and the surface integral $\to 0$ at ∞, so $V(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{\mathbf{E}(\mathbf{r}')\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'$, as before. You can also *check* the result, by computing its gradient—but it's not easy.]

Problem 5.51

(a) For uniform $\mathbf{B}$, $\int_0^{\mathbf{r}}(\mathbf{B}\times d\mathbf{l}) = \mathbf{B}\times\int_0^{\mathbf{r}} d\mathbf{l} = \boxed{\mathbf{B}\times\mathbf{r}} \neq \mathbf{A} = -\frac{1}{2}(\mathbf{B}\times\mathbf{r})$.

(b) $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$, so $\oint \mathbf{B}\times d\mathbf{l} = \left(\frac{\mu_0 I}{2\pi a}\,\hat{\mathbf{s}} - \frac{\mu_0 I}{2\pi b}\,\hat{\mathbf{s}}\right)w = \boxed{\frac{\mu_0 I w}{2\pi}\left(\frac{1}{a} - \frac{1}{b}\right)\hat{\mathbf{s}} \neq 0.}$

(c) $\mathbf{A} = -\mathbf{r}\times\mathbf{B}\int_0^1 \lambda\,d\lambda = \boxed{-\frac{1}{2}(\mathbf{r}\times\mathbf{B}).}$

(d) $\mathbf{B} = \frac{\mu_0 I}{2\pi s}\,\hat{\boldsymbol{\phi}}$; $\mathbf{B}(\lambda\mathbf{r}) = \frac{\mu_0 I}{2\pi\lambda s}\,\hat{\boldsymbol{\phi}}$; $\mathbf{A} = -\frac{\mu_0 I}{2\pi s}(\mathbf{r}\times\hat{\boldsymbol{\phi}})\int_0^1 \lambda\frac{1}{\lambda}\,d\lambda = -\frac{\mu_0 I}{2\pi s}(\mathbf{r}\times\hat{\boldsymbol{\phi}})$. But $\mathbf{r}$ here is the vector from the origin—in cylindrical coordinates $\mathbf{r} = s\,\hat{\mathbf{s}} + z\,\hat{\mathbf{z}}$. So $\mathbf{A} = -\frac{\mu_0 I}{2\pi s}\left[s(\hat{\mathbf{s}}\times\hat{\boldsymbol{\phi}}) + z(\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}})\right]$, and $(\hat{\mathbf{s}}\times\hat{\boldsymbol{\phi}}) = \hat{\mathbf{z}}$, $(\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}}) = -\hat{\mathbf{s}}$. So $\boxed{\mathbf{A} = \frac{\mu_0 I}{2\pi s}(z\,\hat{\mathbf{s}} - s\,\hat{\mathbf{z}}.}$

The examples in (c) and (d) happen to be divergenceless, but this is not the case in general. For (letting $\mathbf{L} \equiv \int_0^1 \lambda\mathbf{B}(\lambda\mathbf{r})\,d\lambda$, for short) $\boldsymbol{\nabla}\cdot\mathbf{A} = -\boldsymbol{\nabla}\cdot(\mathbf{r}\times\mathbf{L}) = -[\mathbf{L}\cdot(\boldsymbol{\nabla}\times\mathbf{r}) - \mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{L})] = \mathbf{r}\cdot(\boldsymbol{\nabla}\times\mathbf{L})$, and $\boldsymbol{\nabla}\times\mathbf{L} = \int_0^1 \lambda[\boldsymbol{\nabla}\times\mathbf{B}(\lambda\mathbf{r})]\,d\lambda = \int_0^1 \lambda^2[\boldsymbol{\nabla}_\lambda\times\mathbf{B}(\lambda\mathbf{r})]\,d\lambda = \mu_0\int_0^1 \lambda^2\mathbf{J}(\lambda\mathbf{r})\,d\lambda$, so $\boldsymbol{\nabla}\cdot\mathbf{A} = \mu_0\mathbf{r}\cdot\int_0^1 \lambda^2\mathbf{J}(\lambda\mathbf{r})\,d\lambda$, and it vanishes in regions where $\mathbf{J} = 0$ (which is why the examples in (c) and (d) were divergenceless). To construct an explicit counterexample, we need the field at a point where $\mathbf{J} \neq 0$—say, *inside* a wire with uniform current.

Here Ampére's law gives $B\,2\pi s = \mu_0 I_{\text{enc}} = \mu_0 J \pi s^2 \Rightarrow \mathbf{B} = \frac{\mu_0 J}{2} s\,\hat{\boldsymbol{\phi}}$, so

$$\begin{aligned}
\mathbf{A} &= -\mathbf{r} \times \int_0^1 \lambda \left(\frac{\mu_0 J}{2}\right) \lambda s\,\hat{\boldsymbol{\phi}}\,d\lambda = -\frac{\mu_0 J}{6} s(\mathbf{r} \times \hat{\boldsymbol{\phi}}) = \frac{\mu_0 J s}{6}(z\,\hat{\mathbf{s}} - s\hat{\mathbf{z}}). \\
\boldsymbol{\nabla} \cdot \mathbf{A} &= \frac{\mu_0 J}{6}\left[\frac{1}{s}\frac{\partial}{\partial s}(s^2 z) + \frac{\partial}{\partial z}(-s^2)\right] = \frac{\mu_0 J}{6}\left(\frac{1}{s} 2sz\right) = \frac{\mu_0 J z}{3} \neq 0.
\end{aligned}$$

Conclusion: $\boxed{\text{(ii) does } \textit{not} \text{ automatically yield } \boldsymbol{\nabla} \cdot \mathbf{A} = 0.}$

Problem 5.52

(a) Exploit the analogy with the electrical case:

$$\begin{aligned}
\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}] \quad \text{(Eq. 3.104)} \ = -\boldsymbol{\nabla} V, \quad \text{with } V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} \quad \text{(Eq. 3.102)}. \\
\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}] \quad \text{(Eq. 5.87)} \ = -\boldsymbol{\nabla} U, \ \text{(Eq. 5.65)}.
\end{aligned}$$

Evidently the prescription is $\mathbf{p}/\epsilon_0 \to \mu_0 \mathbf{m}$: $\boxed{U(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\cdot\hat{\mathbf{r}}}{r^2}.}$

(b) Comparing Eqs. 5.67 and 5.85, the dipole moment of the shell is $\mathbf{m} = (4\pi/3)\omega\sigma R^4\,\hat{\mathbf{z}}$ (which we also got in Prob. 5.36). Using the result of (a), then, $\boxed{U(\mathbf{r}) = \frac{\mu_0 \omega\sigma R^4}{3}\frac{\cos\theta}{r^2}}$ for $r > R$.

Inside the shell, the field is uniform (Eq. 5.38): $\mathbf{B} = \frac{2}{3}\mu_0\sigma\omega R\,\hat{\mathbf{z}}$, so $U(\mathbf{r}) = -\frac{2}{3}\mu_0\sigma\omega R z + \text{constant}$. We may as well pick the constant to be zero, so $\boxed{U(\mathbf{r}) = -\frac{2}{3}\mu_0\sigma\omega R r\cos\theta}$ for $r < R$.

[Notice that $U(\mathbf{r})$ is *not continuous* at the surface ($r = R$): $U_{\text{in}}(R) = -\frac{2}{3}\mu_0\sigma\omega R^2\cos\theta \neq U_{\text{out}}(R) = \frac{1}{3}\mu_0\sigma\omega R^2\cos\theta$. As I warned you on p. 236: if you insist on using magnetic scalar potentials, keep away from places where there is current!]

(c)

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0\omega Q}{4\pi R}\left[\left(1 - \frac{3r^2}{5R^2}\right)\cos\theta\,\hat{\mathbf{r}} - \left(1 - \frac{6r^2}{5R^2}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right] = -\boldsymbol{\nabla} U = -\frac{\partial U}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial U}{\partial\theta}\hat{\boldsymbol{\theta}} - \frac{1}{r\sin\theta}\frac{\partial U}{\partial\phi}\hat{\boldsymbol{\phi}}. \\
\frac{\partial U}{\partial\phi} &= 0 \Rightarrow U(r,\theta,\phi) = U(r,\theta). \\
\frac{1}{r}\frac{\partial U}{\partial\theta} &= \left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1 - \frac{6r^2}{5R^2}\right)\sin\theta \Rightarrow U(r,\theta) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1 - \frac{6r^2}{5R^2}\right) r\cos\theta + f(r). \\
\frac{\partial U}{\partial r} &= -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1 - \frac{3r^2}{5R^2}\right)\cos\theta \Rightarrow U(r,\theta) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(r - \frac{r^3}{5R^2}\right)\cos\theta + g(\theta).
\end{aligned}$$

Equating the two expressions:

$$-\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1 - \frac{6r^2}{5R^2}\right) r\cos\theta + f(r) = -\left(\frac{\mu_0\omega Q}{4\pi R}\right)\left(1 - \frac{r^2}{5R^2}\right) r\cos\theta + g(\theta),$$

or

$$\left(\frac{\mu_0\omega Q}{4\pi R^3}\right) r^3\cos\theta + f(r) = g(\theta).$$

But there is no way to write $r^3\cos\theta$ as the sum of a function of θ and a function of r, so we're stuck. The reason is that you can't *have* a scalar magnetic potential in a region where the current is nonzero.

Problem 5.53

(a) $\boldsymbol{\nabla}\cdot\mathbf{B}=0,\ \boldsymbol{\nabla}\times\mathbf{B}=\mu_0\mathbf{J}$, and $\boldsymbol{\nabla}\cdot\mathbf{A}=0,\ \boldsymbol{\nabla}\times\mathbf{A}=\mathbf{B}\ \Rightarrow \mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}}{\mathscr{r}}\,d\tau'$, so

$\boldsymbol{\nabla}\cdot\mathbf{A}=0,\ \boldsymbol{\nabla}\times\mathbf{A}=\mathbf{B}$, and $\boldsymbol{\nabla}\cdot\mathbf{W}=0$ (we'll choose it so), $\boldsymbol{\nabla}\times\mathbf{W}=\mathbf{A}\ \Rightarrow \boxed{\mathbf{W}=\frac{1}{4\pi}\int\frac{\mathbf{B}}{\mathscr{r}}\,d\tau'.}$

(b) $\mathbf{W}$ will be proportional to $\mathbf{B}$ and to two factors of $\mathbf{r}$ (since differentiating *twice* must recover $\mathbf{B}$), so I'll try something of the form $\mathbf{W}=\alpha\mathbf{r}(\mathbf{r}\cdot\mathbf{B})+\beta r^2\mathbf{B}$, and see if I can pick the constants α and β in such a way that $\boldsymbol{\nabla}\cdot\mathbf{W}=0$ and $\boldsymbol{\nabla}\times\mathbf{W}=\mathbf{A}$.

$\boldsymbol{\nabla}\cdot\mathbf{W}=\alpha\left[(\mathbf{r}\cdot\mathbf{B})(\boldsymbol{\nabla}\cdot\mathbf{r})+\mathbf{r}\cdot\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{B})\right]+\beta\left[r^2(\boldsymbol{\nabla}\cdot\mathbf{B})+\mathbf{B}\cdot\boldsymbol{\nabla}(r^2)\right]$. $\boldsymbol{\nabla}\mathbf{r}=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=1+1+1=3$;

$\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{B})=\mathbf{r}\times(\boldsymbol{\nabla}\times\mathbf{B})+\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{r})+(\mathbf{r}\cdot\boldsymbol{\nabla})\mathbf{B}+(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{r}$; but $\mathbf{B}$ is constant, so all derivatives of $\mathbf{B}$ vanish, and $\boldsymbol{\nabla}\times\mathbf{r}=0$ (Prob. 1.62), so

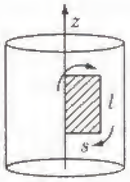
$\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{B})=(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{r}=\left(B_x\frac{\partial}{\partial x}+B_y\frac{\partial}{\partial y}+B_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}})=B_x\,\hat{\mathbf{x}}+B_y\,\hat{\mathbf{y}}+B_z\,\hat{\mathbf{z}}=\mathbf{B}$;

$\boldsymbol{\nabla}(r^2)=\left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)=2x\,\hat{\mathbf{x}}+2y\,\hat{\mathbf{y}}+2z\,\hat{\mathbf{z}}=2\mathbf{r}$. So

$\boldsymbol{\nabla}\cdot\mathbf{W}=\alpha\left[3(\mathbf{r}\cdot\mathbf{B})+(\mathbf{r}\cdot\mathbf{B})\right]+\beta\left[0+2(\mathbf{r}\cdot\mathbf{B})\right]=2(\mathbf{r}\cdot\mathbf{B})(2\alpha+\beta)$, which is zero if $2\alpha+\beta=0$.

$\boldsymbol{\nabla}\times\mathbf{W}=\alpha\left[(\mathbf{r}\cdot\mathbf{B})(\boldsymbol{\nabla}\times\mathbf{r})-\mathbf{r}\times\boldsymbol{\nabla}(\mathbf{r}\cdot\mathbf{B})\right]+\beta\left[r^2(\boldsymbol{\nabla}\times\mathbf{B})-\mathbf{B}\times\boldsymbol{\nabla}(r^2)\right]=\alpha\left[0-(\mathbf{r}\times\mathbf{B})\right]+\beta\left[0-2(\mathbf{B}\times\mathbf{r})\right]$
$=-(\mathbf{r}\times\mathbf{B})(\alpha-2\beta)=-\frac{1}{2}(\mathbf{r}\times\mathbf{B})$ (Prob. 5.24). So we want $\alpha-2\beta=1/2$. Evidently $\alpha-2(-2\alpha)=5\alpha=1/2$, or $\alpha=1/10;\beta=-2\alpha=-1/5$. *Conclusion:* $\boxed{\mathbf{W}=\frac{1}{10}\left[\mathbf{r}(\mathbf{r}\cdot\mathbf{B})-2r^2\mathbf{B}\right].}$ (But this is certainly not unique.)

(c) $\boldsymbol{\nabla}\times\mathbf{W}=\mathbf{A}\Rightarrow\int(\boldsymbol{\nabla}\times\mathbf{W})\cdot d\mathbf{a}=\int\mathbf{A}\cdot d\mathbf{a}$. Or $\oint\mathbf{W}\cdot d\mathbf{l}=\int\mathbf{A}\cdot d\mathbf{a}$. Integrate around the amperian loop shown, taking $\mathbf{W}$ to point parallel to the axis, and choosing $\mathbf{W}=0$ on the axis:



$-Wl=\int_0^s\left(\frac{\mu_0 nI}{2}\right)l\bar{s}\,d\bar{s}=\frac{\mu_0 nI}{2}\frac{s^2 l}{2}$ (using Eq. 5.70 for $\mathbf{A}$).

$\boxed{\mathbf{W}=-\frac{\mu_0 nIs^2}{4}\,\hat{\mathbf{z}}}$ $(s<R)$.

For $s>R$, $-Wl=\frac{\mu_0 nIR^2 l}{4}+\int_R^s\left(\frac{\mu_0 nI}{2}\right)\frac{R^2}{\bar{s}}l\,d\bar{s}=\frac{\mu_0 nIR^2 l}{4}+\frac{\mu_0 nIR^2 l}{2}\ln(s/R)$;

$$\boxed{\mathbf{W}=-\frac{\mu_0 nIR^2}{4}\left[1+2\ln(s/R)\right]\hat{\mathbf{z}}}\ (s>R).$$

Problem 5.54

Apply the divergence theorem to the function $[\mathbf{U}\times(\boldsymbol{\nabla}\times\mathbf{V})]$, noting (from the product rule) that $\boldsymbol{\nabla}\cdot[\mathbf{U}\times(\boldsymbol{\nabla}\times\mathbf{V})]=(\boldsymbol{\nabla}\times\mathbf{V})\cdot(\boldsymbol{\nabla}\times\mathbf{U})-\mathbf{U}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{V})]$:

$$\int\boldsymbol{\nabla}\cdot[\mathbf{U}\times(\boldsymbol{\nabla}\times\mathbf{V})]\,d\tau=\int\{(\boldsymbol{\nabla}\times\mathbf{V})\cdot(\boldsymbol{\nabla}\times\mathbf{U})-\mathbf{U}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{V})]\}\,d\tau=\oint[\mathbf{U}\times(\boldsymbol{\nabla}\times\mathbf{V})]\cdot d\mathbf{a}.$$

As always, suppose we have *two* solutions, $\mathbf{B}_1$ (and $\mathbf{A}_1$) and $\mathbf{B}_2$ (and $\mathbf{A}_2$). Define $\mathbf{B}_3\equiv\mathbf{B}_2-\mathbf{B}_1$ (and $\mathbf{A}_3\equiv\mathbf{A}_2-\mathbf{A}_1$), so that $\boldsymbol{\nabla}\times\mathbf{A}_3=\mathbf{B}_3$ and $\boldsymbol{\nabla}\times\mathbf{B}_3=\boldsymbol{\nabla}\times\mathbf{B}_1-\boldsymbol{\nabla}\times\mathbf{B}_2=\mu_0\mathbf{J}-\mu_0\mathbf{J}=0$. Set $\mathbf{U}=\mathbf{V}=\mathbf{A}_3$ in the above identity:

$$\int \{(\nabla \times \mathbf{A}_3) \cdot (\nabla \times \mathbf{A}_3) - \mathbf{A}_3 \cdot [\nabla \times (\nabla \times \mathbf{A}_3)]\}\, d\tau = \int \{(\mathbf{B}_3) \cdot (\mathbf{B}_3) - \mathbf{A}_3 \cdot [\nabla \times \mathbf{B}_3]\}\, d\tau = \int (B_3)^2\, d\tau$$

$= \oint [\mathbf{A}_3 \times (\nabla \times \mathbf{A}_3)] \cdot d\mathbf{a} = \oint (\mathbf{A}_3 \times \mathbf{B}_3) \cdot d\mathbf{a}$. But either **A** is specified (in which case $\mathbf{A}_3 = 0$), or else **B** is specified (in which case $\mathbf{B}_3 = 0$), at the surface. In either case $\oint (\mathbf{A}_3 \times \mathbf{B}_3) \cdot d\mathbf{a} = 0$. So $\int (B_3)^2\, d\tau = 0$, and hence $\mathbf{B}_1 = \mathbf{B}_2$. qed

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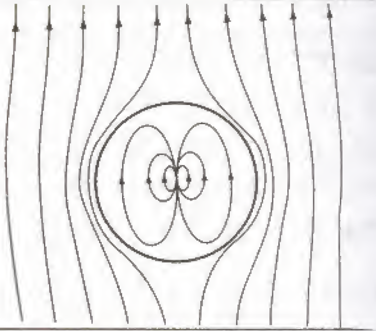
Problem 5.55

From Eq. 5.86, $\mathbf{B}_{\text{tot}} = B_0\,\hat{\mathbf{z}} - \frac{\mu_0 m_0}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})$. Therefore $\mathbf{B} \cdot \hat{\mathbf{r}} = B_0(\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}) - \frac{\mu_0 m_0}{4\pi r^3} 2\cos\theta = \left(B_0 - \frac{\mu_0 m_0}{2\pi r^3}\right)\cos\theta$. This is zero, for all θ, when $r = R$, given by $B_0 = \frac{\mu_0 m_0}{2\pi R^3}$, or $\boxed{R = \left(\frac{\mu_0 m_0}{2\pi B_0}\right)^{1/3}.}$ Evidently no field lines cross this sphere.



Problem 5.56

(a) $I = \frac{Q}{(2\pi/\omega)} = \frac{Q\omega}{2\pi}$; $a = \pi R^2$; $\mathbf{m} = \frac{Q\omega}{2\pi}\pi R^2\,\hat{\mathbf{z}} = \frac{Q}{2}\omega R^2\,\hat{\mathbf{z}}$. $L = RMv = M\omega R^2; \mathbf{L} = M\omega R^2\,\hat{\mathbf{z}}$.

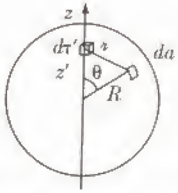
$\frac{m}{L} = \frac{Q}{2}\frac{\omega R^2}{M\omega R^2} = \frac{Q}{2M}$. $\boxed{\mathbf{m} = \left(\frac{Q}{2M}\right)\mathbf{L},}$ and the gyromagnetic ratio is $\boxed{g = \frac{Q}{2M}.}$

(b) Because g is *independent of* R, the same ratio applies to all "donuts", and hence to the entire sphere (or any other figure of revolution): $\boxed{g = \frac{Q}{2M}.}$

(c) $m = \frac{e}{2m}\frac{\hbar}{2} = \frac{e\hbar}{4m} = \frac{(1.60 \times 10^{-19})(1.05 \times 10^{-34})}{4(9.11 \times 10^{-31})} = \boxed{4.61 \times 10^{-24}\,\mathrm{A\,m^2}.}$

Problem 5.57

(a) $\mathbf{B}_{\text{ave}} = \frac{1}{(3/4)\pi R^3}\int \mathbf{B}\, d\tau = \frac{3}{4\pi R^3}\int (\nabla \times \mathbf{A})\, d\tau = -\frac{3}{4\pi R^3}\oint \mathbf{A} \times d\mathbf{a} = -\frac{3}{4\pi R^3}\frac{\mu_0}{4\pi}\oint \left\{\int \frac{\mathbf{J}}{\mathscr{r}}\, d\tau'\right\} \times d\mathbf{a} = -\frac{3\mu_0}{(4\pi)^2 R^3}\int \mathbf{J} \times \left\{\oint \frac{1}{\mathscr{r}}\, d\mathbf{a}\right\} d\tau'$. Note that **J** depends on the source point $\mathbf{r}'$, not on the field point $\mathbf{r}$. To do the surface integral, choose the (x, y, z) coordinates so that $\mathbf{r}'$ lies on the z axis (see diagram). Then $\mathscr{r} = \sqrt{R^2 + (z')^2 - 2Rz'\cos\theta}$, while $d\mathbf{a} = R^2 \sin\theta\, d\theta\, d\phi\,\hat{\mathbf{r}}$. By symmetry, the x and y components must integrate to zero; since the z component of $\hat{\mathbf{r}}$ is $\cos\theta$, we have



$$\begin{aligned}
\oint \frac{1}{\mathscr{r}}\, d\mathbf{a} &= \hat{\mathbf{z}} \int \frac{\cos\theta}{\sqrt{R^2+(z')^2-2Rz'\cos\theta}} R^2 \sin\theta \, d\theta \, d\phi = 2\pi R^2 \, \hat{\mathbf{z}} \int_0^\pi \frac{\cos\theta \sin\theta}{\sqrt{R^2+(z')^2-2Rz'\cos\theta}} \, d\theta. \\
&\ \text{Let } u \equiv \cos\theta, \text{ so } du = -\sin\theta \, d\theta. \\
&= 2\pi R^2 \, \hat{\mathbf{z}} \int_{-1}^{1} \frac{u}{\sqrt{R^2+(z')^2-2Rz'u}} \, du \\
&= 2\pi R^2 \, \hat{\mathbf{z}} \left\{ -\frac{2\left[2(R^2+(z')^2)+2Rz'u\right]}{3(2Rz')^2} \sqrt{R^2+(z')^2-2Rz'u} \right\} \Bigg|_{-1}^{1} \\
&= -\frac{2\pi R^2 \, \hat{\mathbf{z}}}{3(Rz')^2} \left\{ \left[R^2+(z')^2+Rz'\right] \sqrt{R^2+(z')^2-2Rz'} - \left[R^2+(z')^2-Rz'\right] \sqrt{R^2+(z')^2+2Rz'} \right\} \\
&= -\left[\frac{2\pi}{3(z')^2} \hat{\mathbf{z}}\right] \left\{ \left[R^2+(z')^2+Rz'\right] |R-z'| - \left[R^2+(z')^2-Rz'\right](R+z') \right\} \\
&= \left\{ \begin{array}{ll} \dfrac{4\pi}{3} z' \, \hat{\mathbf{z}} = \dfrac{4\pi}{3} \mathbf{r}', & (r' < R); \\[2ex] \dfrac{4\pi R^3}{3(z')^2} \, \hat{\mathbf{z}} = \dfrac{4\pi}{3} \dfrac{R^3}{(r')^3} \mathbf{r}', & (r' > R). \end{array} \right\}
\end{aligned}$$

For now we want $r' < R$, so $\mathbf{B}_{\text{ave}} = -\dfrac{3\mu_0}{(4\pi)^2 R^3} \dfrac{4\pi}{3} \displaystyle\int (\mathbf{J} \times \mathbf{r}') \, d\tau' = -\dfrac{\mu_0}{4\pi R^3} \displaystyle\int (\mathbf{J} \times \mathbf{r}') \, d\tau'$. Now $\mathbf{m} = \frac{1}{2} \int (\mathbf{r} \times \mathbf{J}) \, d\tau$ (Eq. 5.91), so $\mathbf{B}_{\text{ave}} = \dfrac{\mu_0}{4\pi} \dfrac{2\mathbf{m}}{R^3}$. qed

(b) This time $r' > R$, so $\mathbf{B}_{\text{ave}} = -\dfrac{3\mu_0}{(4\pi)^2 R^3} \dfrac{4\pi}{3} R^3 \displaystyle\int \left(\mathbf{J} \times \frac{\mathbf{r}'}{(r')^3} \right) d\tau' = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\mathbf{J} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$, where $\boldsymbol{\mathscr{r}}$ now goes from the source point to the *center* ($\boldsymbol{\mathscr{r}} = -\mathbf{r}'$). Thus $\mathbf{B}_{\text{ave}} = \mathbf{B}_{\text{cen}}$. qed

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Problem 5.58

(a) Problem 5.51 gives the dipole moment of a shell: $\mathbf{m} = \dfrac{4\pi}{3} \sigma \omega R^4 \, \hat{\mathbf{z}}$. Let $R \to r, \sigma \to \rho \, dr$, and integrate:

$$\mathbf{m} = \frac{4\pi}{3} \omega \rho \, \hat{\mathbf{z}} \int_0^R r^4 \, dr = \frac{4\pi}{3} \omega \rho \frac{R^5}{5} \, \hat{\mathbf{z}}. \quad \text{But } \rho = \frac{Q}{(4/3)\pi R^3}, \text{ so } \boxed{\mathbf{m} = \frac{1}{5} Q \omega R^2 \, \hat{\mathbf{z}}.}$$

(b) $\mathbf{B}_{\text{ave}} = \dfrac{\mu_0}{4\pi} \dfrac{2\mathbf{m}}{R^3} = \boxed{\dfrac{\mu_0}{4\pi} \dfrac{2Q\omega}{5R} \, \hat{\mathbf{z}}.}$

(c) $\mathbf{A} \cong \dfrac{\mu_0}{4\pi} \dfrac{m \sin\theta}{r^2} \, \hat{\boldsymbol{\phi}} = \boxed{\dfrac{\mu_0}{4\pi} \dfrac{Q\omega R^2}{5} \dfrac{\sin\theta}{r^2} \, \hat{\boldsymbol{\phi}}.}$

(d) Use Eq. 5.67, with $R \to \bar{r}, \sigma \to \rho \, d\bar{r}$, and integrate:

$$\mathbf{A} = \frac{\mu_0 \omega \rho}{3} \frac{\sin\theta}{r^2} \, \hat{\boldsymbol{\phi}} \int_0^R \bar{r}^4 \, d\bar{r} = \frac{\mu_0 \omega}{3} \frac{3Q}{4\pi R^3} \frac{\sin\theta}{r^2} \frac{R^5}{5} \, \hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0}{4\pi} \frac{Q\omega R^2}{5} \frac{\sin\theta}{r^2} \, \hat{\boldsymbol{\phi}}.}$$

This is *identical* to (c); evidently the field is *pure* dipole, for points outside the sphere.

(e) According to Prob. 5.29, the field is $\mathbf{B} = \dfrac{\mu_0 \omega Q}{4\pi R} \left[\left(1 - \dfrac{3r^2}{5R^2}\right) \cos\theta \, \hat{\mathbf{r}} - \left(1 - \dfrac{6r^2}{5R^2}\right) \sin\theta \, \hat{\boldsymbol{\theta}} \right]$. The average obviously points in the z direction, so take the z component of $\hat{\mathbf{r}}$ ($\cos\theta$) and $\hat{\boldsymbol{\theta}}$ ($-\sin\theta$):

$$\begin{aligned}
B_{\text{ave}} &= \frac{\mu_0\omega Q}{4\pi R}\frac{1}{(4/3)\pi R^3}\int\left[\left(1-\frac{3r^2}{5R^2}\right)\cos^2\theta+\left(1-\frac{6r^2}{5R^2}\right)\sin^2\theta\right]r^2\sin\theta\,dr\,d\theta\,d\phi\\
&= \frac{3\mu_0\omega Q}{(4\pi R^2)^2}2\pi\int_0^\pi\left[\left(\frac{r^3}{3}-\frac{3}{5}\frac{R^5}{5R^2}\right)\cos^2\theta+\left(\frac{R^3}{3}-\frac{6}{5}\frac{R^5}{5R^2}\right)\sin^2\theta\right]\sin\theta\,d\theta\\
&= \frac{3\mu_0\omega Q}{8\pi R^4}R^3\int_0^\pi\left(\frac{16}{75}\cos^2\theta+\frac{7}{75}\sin^2\theta\right)\sin\theta\,d\theta=\frac{3\mu_0\omega Q}{8\pi R}\frac{1}{75}\int_0^\pi(7+9\cos^2\theta)\sin\theta\,d\theta\\
&= \frac{\mu_0\omega Q}{200\pi R}\left(-7\cos\theta-3\cos^3\theta\right)\Big|_0^\pi=\frac{\mu_0\omega Q}{200\pi R}(20)=\frac{\mu_0\omega Q}{10\pi R}\ \text{(same as (b)).}\ \checkmark
\end{aligned}$$

Problem 5.59

The issue (and the integral) is identical to the one in Prob. 3.42. The resolution (as before) is to regard Eq. 5.87 as correct outside an infinitesimal sphere centered at the dipole. *Inside* this sphere the field is a delta-function, $\mathbf{A}\delta^3(\mathbf{r})$, with $\mathbf{A}$ selected so as to make the average field consistent with Prob. 5.57:

$$\mathbf{B}_{\text{ave}}=\frac{1}{(4/3)\pi R^3}\int\mathbf{A}\delta^3(\mathbf{r})\,d\tau=\frac{3}{4\pi R^3}\mathbf{A}=\frac{\mu_0}{4\pi}\frac{2\mathbf{m}}{R^3}\Rightarrow\mathbf{A}=\frac{2\mu_0\mathbf{m}}{3}.\ \text{The added term is}\ \boxed{\frac{2\mu_0}{3}\mathbf{m}\delta^3(\mathbf{r}).}$$

Problem 5.60

(a) $I\,d\mathbf{l}\to\mathbf{J}\,d\tau$, so $\boxed{\mathbf{A}=\frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int(r')^nP_n(\cos\theta)\mathbf{J}\,d\tau.}$

(b) $\mathbf{A}_{\text{mon}}=\dfrac{\mu_0}{4\pi r}\displaystyle\int\mathbf{J}\,d\tau=\dfrac{\mu_0}{4\pi r}\dfrac{d\mathbf{p}}{dt}$ (Prob. 5.7), where $\mathbf{p}$ is the total electric dipole moment. In magnetostatics, $\mathbf{p}$ is constant, so $d\mathbf{p}/dt=0$, and hence $\mathbf{A}_{\text{mon}}=0$. qed

(c) $\mathbf{m}=I\mathbf{a}=\frac{1}{2}I\oint(\mathbf{r}\times d\mathbf{l})\to\mathbf{m}=\frac{1}{2}\int(\mathbf{r}\times\mathbf{J})\,d\tau.$ qed

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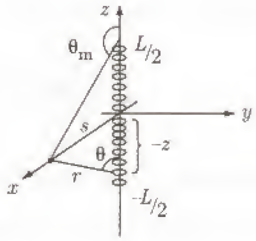
Problem 5.61

For a dipole at the origin and a field point in the xz plane ($\phi=0$), we have

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}})=\frac{\mu_0}{4\pi}\frac{m}{r^3}[2\cos\theta(\sin\theta\,\hat{\mathbf{x}}+\cos\theta\,\hat{\mathbf{z}})+\sin\theta(\cos\theta\,\hat{\mathbf{x}}-\sin\theta\,\hat{\mathbf{z}})]\\
&= \frac{\mu_0}{4\pi}\frac{m}{r^3}[3\sin\theta\cos\theta\,\hat{\mathbf{x}}+(2\cos^2\theta-\sin^2\theta)\,\hat{\mathbf{z}}].
\end{aligned}$$

Here we have a *stack* of such dipoles, running from $z=-L/2$ to $z=+L/2$. Put the field point at s on the x axis. The $\hat{\mathbf{x}}$ components cancel (because of symmetrically placed dipoles above and below $z=0$), leaving $\mathbf{B}=\dfrac{\mu_0}{4\pi}2\mathcal{M}\,\hat{\mathbf{z}}\displaystyle\int_0^{L/2}\frac{(3\cos^2\theta-1)}{r^3}\,dz$, where $\mathcal{M}$ is the dipole moment per unit length: $m=I\pi R^2=(\sigma vh)\pi R^2=\sigma\omega R\pi R^2h\Rightarrow\mathcal{M}=\dfrac{m}{h}=\pi\sigma\omega R^3$. Now $\sin\theta=\dfrac{s}{r}$, so $\dfrac{1}{r^3}=\dfrac{\sin^3\theta}{s^3}$; $z=-s\cot\theta\ \Rightarrow dz=\dfrac{s}{\sin^2\theta}\,d\theta$. Therefore



$$\begin{aligned}\mathbf{B} &= \frac{\mu_0}{2\pi}(\pi\sigma\omega R^3)\,\hat{\mathbf{z}}\int_{\pi/2}^{\theta_m}(3\cos^2\theta-1)\frac{\sin^3\theta}{s^3}\frac{s}{\sin^2\theta}\,d\theta = \frac{\mu_0\sigma\omega R^3}{2s^2}\,\hat{\mathbf{z}}\int_{\pi/2}^{\theta_m}(3\cos^2\theta-1)\sin\theta\,d\theta \\ &= \frac{\mu_0\sigma\omega R^3}{2s^2}\,\hat{\mathbf{z}}\left(-\cos^3\theta+\cos\theta\right)\Big|_{\pi/2}^{\theta_m} = \frac{\mu_0\sigma\omega R^3}{2s^2}\cos\theta_m\left(1-\cos^2\theta_m\right)\,\hat{\mathbf{z}} = \frac{\mu_0\sigma\omega R^3}{2s^2}\cos\theta_m\sin^2\theta_m\,\hat{\mathbf{z}}.\end{aligned}$$

But $\sin\theta_m = \dfrac{s}{\sqrt{s^2+(L/2)^2}}$, and $\cos\theta_m = \dfrac{-(L/2)}{\sqrt{s^2+(L/2)^2}}$, so $\boxed{\mathbf{B} = -\dfrac{\mu_0\sigma\omega R^3 L}{4[s^2+(L/2)^2]^{3/2}}\,\hat{\mathbf{z}}.}$